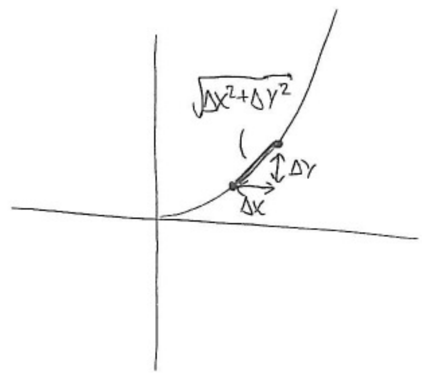
Here are brief solutions to the problems discussed in the first recitation meeting. Don't worry if you had trouble with any or all of them – it's just a warm-up to get you thinking about calculus again, and these were pretty tricky.

1. *What's the length of the curve $y = x^2$ from $x = 2$ to $x = 3$? (warning: actually evaluating the integral might be the hardest part) See if you can derive the formula for the arc length of a curve, whether you remember it or not.*

   First let's derive the formula for the length of an arc. Draw the graph of the function:



   To compute the length, we'll break the curve into lots of very small pieces, each of which is almost straight. Consider the piece between x and $x + \Delta x$. What's its length? By the Pythagorean theorem, this is $\Delta \ell = \sqrt{\Delta x^2 + \Delta y^2}$.

   We can rewrite this as

$$\Delta \ell = \sqrt{\Delta x^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

   The total length of the arc is now given by adding all of these together, i.e. taking the integral:

$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

   This is the formula you saw in 18.01 and perhaps not since.

   Now we can use this formula to answer the original question. Our function is $y = x^2$, and $\frac{dy}{dx} = 2x$. So we need to evaluate the integral:

$$\ell = \int_2^3 \sqrt{1 + 4x^2}\, dx.$$

   This is a tricky integral – you might not have encountered it before, and evaluating it will give you a real workout for your methods of integration. The easiest way to do it (barring looking it up in the integration table in the back of the book or using Mathematica) is to use some identities involving the "hyperbolic functions" $\cosh(x)$

and $\sinh(x)$. If you haven't heard about these functions before, you can read the first couple paragraphs of the Wiki article or check a calculus textbook. They pop up surprisingly often when doing integrals or solving differential equations, and I think it's worth remembering a little bit about them.

In our case the relevant identity is

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + C.$$

This function is the inverse function of $\sinh(x)$. Since $\sinh(x) = \frac{e^x - e^{-x}}{2}$, you can actually use the quadratic formula to find that $\sinh^{-1}(x) = \log(x \pm \sqrt{x^2+1})$ (I went through this in section). You can double-check the integral that I just claimed by using the chain rule to compute the derivative of this expression. NB: whenever I write "log" without specifying a base, I mean the natural logarithm.

Let's first evaluate the indefinite integral $\int \sqrt{1+x^2}\, dx$. The strategy is to cleverly integrate by parts.

Take "u"$= \sqrt{1+x^2}$ and "v"$= x$. The formula for integration by parts gives you

$$\begin{aligned}\int \sqrt{1+x^2}\, dx &= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\, dx + C \\ &= x\sqrt{1+x^2} - \left(\frac{1+x^2}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}}\right) dx + C \\ &= x\sqrt{1+x^2} - \int \sqrt{1+x^2}\, dx + \int \frac{1}{\sqrt{1+x^2}}\, dx + C.\end{aligned}$$

But the second term above is the same thing we started with! It seems that all is lost... but in fact we're almost done. Call the integral in question L. We have obtained

$$\begin{aligned}L &= x\sqrt{1+x^2} - L + \int \frac{1}{\sqrt{1+x^2}}\, dx + C \\ 2L &= x\sqrt{1+x^2} + \int \frac{1}{\sqrt{1+x^2}}\, dx + C\end{aligned}$$

So we can solve for L! Remember from above the the right-hand term gives $\sinh^{-1}(x)$ and we get

$$\int \sqrt{1+x^2}\, dx = \frac{x\sqrt{1+x^2}}{2} + \frac{\sinh^{-1}}{2} + C.$$

If you don't like this $\sinh^{-1}(x)$ thing floating around, you can rewrite this as

$$\int \sqrt{1+x^2}\, dx = \frac{x\sqrt{1+x^2}}{2} + \frac{\log\left(x+\sqrt{1+x^2}\right)}{2}$$

Taking the derivative of both sides, you'll see that they agree. If you don't yet trust your TA, you can check that this agrees with integral #44 in the table at the back of the course textbook.

Now all that's left to finish the problem is to plug in the values for our question (there's also a substitution $u = 2x$, but I'll leave it to you to work out the details). The answer we get is

$$\ell = \frac{6\sqrt{37} - 4\sqrt{17} - \log(4+\sqrt{17}) + \log(6+\sqrt{37})}{4}$$

Phew! In retrospect this was probably a poor choice for the first problem in recitation of the year – sorry!

2. *Define a curve in the plane by $x(t) = t\cos t$, $y(t) = t\sin t$. Sketch the curve. What is the length of the arc it traces when t ranges from 0 to a?*

   We can get a formula for arc length of a parametrized curve using the same method as before:

$$\ell = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

   We have

$$\frac{dx}{dt} = -t\sin t + \cos t, \frac{dy}{dt} = t\cos t + \sin t$$

   Thus the length is

$$\begin{aligned}\ell &= \int_0^a \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2}\, dt \\ &= \int_0^a \sqrt{t^2\sin^2 t + t^2\cos^2 t + \cos^2 t + \sin^2 t + 2t\cos t\sin t - 2t\cos t\sin t}\, dt \\ &= \int_0^a \sqrt{t^2+1}\, dt.\end{aligned}$$

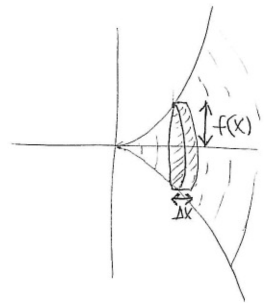
   Lucky for us we just did this integral for the last problem, and we obtain

$$\ell = \frac{a\sqrt{1+a^2}}{2} + \frac{\log(a + \sqrt{a^2+1})}{2}.$$

3. *Now rotate $y = x^2$ for $2 \leq x \leq 3$ about the x-axis to obtain a surface of revolution. What is its volume? Can you compute it in more than one way? What is its surface area? How can you compute the area of a surface of revolution?*

   Volumes of surfaces of revolution is something that you probably encountered in 18.01. We'll do a lot more computing volumes via integration in 18.02. You probably saw two methods, called the "shell method" and the "washer method". If you're like me, you probably don't remember the formulas for either one, so let's derive them. We'll use the same sort of ideas that we did to find the formula for arc length.

   First make slices perpendicular to the x-axis.



I think this one is the "washer method". The slice between x and $x+\Delta x$ is roughly a cylinder of height Δx and radius $f(x)$, so it has volume $\pi f(x)^2\,\Delta x$. Integrating all of these gives a total volume of

$$V = \int_a^b \pi f(x)^2\,dx.$$

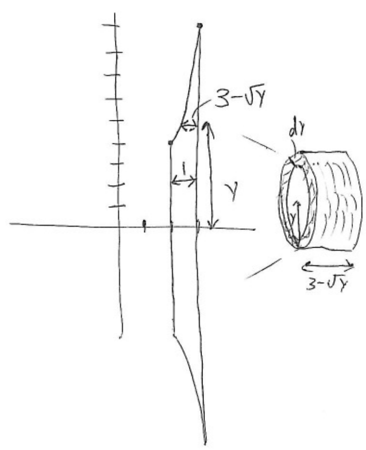
For the surface in question, this is just

$$V = \int_2^3 \pi (x^2)^2\,dx.$$

This time we get a reprieve from doing horrible integrals: this is just

$$V = \pi \int_2^3 x^4\,dx = \pi\left(\frac{x^5}{5}\right)\Bigg|_2^3 = \frac{211\pi}{5}.$$

The other method of finding the volume is to break it up into shells parallel the x-axis (hence "shell method"). Each shell has thickness Δy, radius y, and a height h that we can read off the picture.

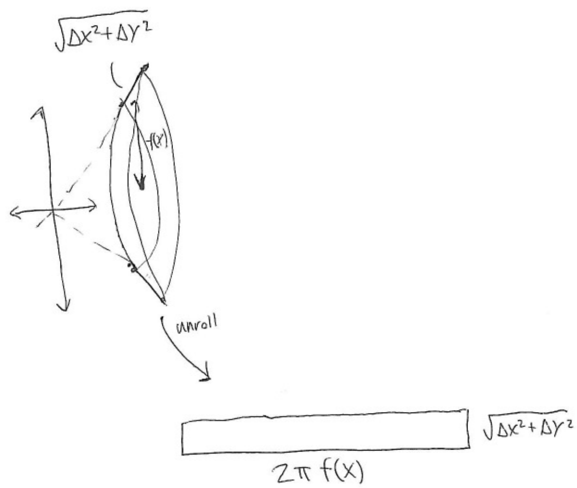


Its volume is therefore $2\pi y h\,dy$. In the example, the height of the slice is $3-\sqrt{y}$ when y is between 4 and 9, and 1 when y is between 0 and 4. So

$$V = \int_0^4 2\pi y(3)\,dy + \int_4^9 2\pi y(3-\sqrt{y})\,dy = 16\pi + \frac{131\pi}{5} = \frac{211\pi}{5},$$

which agrees with the first computation.

The surface area is a bit trickier to work out. Let's figure out the formula as before.



Between x and $x + \Delta x$ there is a region which is roughly cylindrical. Its radius is $f(x)$ and its height is the arc length $\sqrt{\Delta x^2 + \Delta y^2}$. Thus it contributes an area of about $2\pi f(x)\sqrt{\Delta x^2 + \Delta y^2}$. Adding these up as before,

$$A = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

This formula tends to spit out integrals which are quite hard to evaluate. This time it's

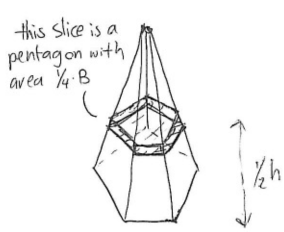
$$A = \int_2^3 2\pi x^2\sqrt{1 + 4x^2}\, dx.$$

You can do it using the $\sinh^{-1}(x)$ as before, but I'm not going to write out all the details this time. Mathematica informs me that it's

$$A = \frac{\pi}{32}\left(438\sqrt{37} - 132\sqrt{17} + \log(4 + \sqrt{17}) - \log(6 + \sqrt{37})\right).$$

4. *The volume of a cone is always $Bh/3$, where B is the area of the base and h is the height, regardless of the shape of the base. Can you prove this? Why doesn't the shape of the base matter?*

At last we come to a more conceptual question with no nasty integrals. To compute the volume of the cone, we're going to cut it up into slices parallel to the base.



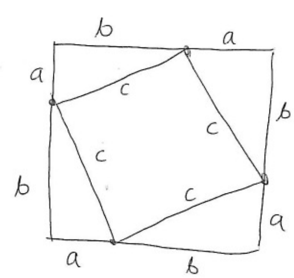
First think about the slice that's halfway up the cone. Its cross section looks the same as the base, but it's scaled down by a factor of 2 in both the x and the y direction. So its area is $B/4$. In general, if we're a fraction s/h up the cone, the corresponding slice is going to have area $(s/h)^2B$, and therefore volume $(s/h)^2B\,ds$. Thus we get

$$V = \int_0^h B\left(\frac{s}{h}\right)^2 ds = \frac{Bh}{3}.$$

This explains the formula for the volumes of various pyramids that you might have encountered the first time you took geometry. And it doesn't depend on the shape of the base because each slice is just some scaled-down version of the base, regardless of its specific form.

5. *What does the Pythagorean theorem state? Can you come up with a proof? What about the law of sines and the law of cosines?*

The Pythagorean theorem states that for a right triangle, $a^2 + b^2 = c^2$, where a and b are the lengths of the legs, and c is the length of the hypotenuse. There are literally hundreds of distinct ways to prove it – President Garfield even came up with one. One strategy that works is to piece several triangles together to give a shape whose area can be computed in terms of a, b, and c in more than one way. Here's one I find memorable:

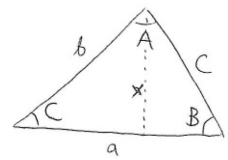


The total area of the square is $(a + b)^2 = a^2 + 2ab + b^2$. We can also compute it is $c^2 + 4 \cdot (ab/2)$, where $ab/2$ is the area of each of the four triangles. Equating these two expressions then yields $a^2 + b^2 = c^2$.

The law of sines states that for a triangle with sides of lengths a, b, and c, and opposing angles A, B, and C, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

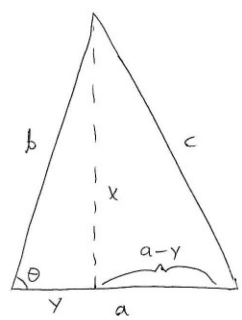
Here's the proof: draw an altitude as in the picture below, and call its length x. This splits the triangle into two right triangles, and we can compute x in two different ways. From the left side, we see that $\sin C = x/b$, so $x = b \sin C$. On the right, we get $\sin B = x = c$, so $x = c \sin B$. Equating these gives $\sin B/b = \sin C/c$. To get the equality with a and A, use the same argument but with one of the other altitudes of the triangle.



The law of cosines says that in the same situation, we have the relation

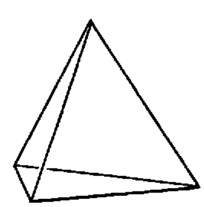
$$a^2 + b^2 = c^2 - 2ab\cos C.$$

If $C = \pi/2 = 90°$, the $\cos C$ term is zero and we get the Pythagorean theorem. To prove this one, start with the same diagram.



We have $x = b\sin\theta$ and $y = b\cos\theta$. Applying the Pythagorean theorem to the triangle on the right side, $(a-y)^2+x^2 = c^2$. Expanding this, $a^2-2ab\cos\theta+b^2\cos^2\theta+b^2\sin^2\theta = c^2$. Applying $\sin^2\theta + \cos^2\theta = 1$ and simplifying, we obtain $a^2 + b^2 - 2ab\cos\theta = c^2$. (Oops, I guess I changed C to θ in the proof).

6, 7 *A* regular tetrahedron *is a polyhedron with four sides, each an equilateral triangle.*



Suppose that the side length of a tetrahedron is a. Can you compute:

- *The surface area?*
- *The height?*
- *The volume?*
- *The angle between two faces?*
- *The radius of an inscribed sphere?*

A Platonic solid *is a polyhedron whose faces are regular polygons, with the same number of faces meeting at each vertex. Can you describe all the Platonic solids? (Hint: there are five types) Pick one of the Platonic solids that isn't the tetrahedron or the cube, and answer the same questions as above.*

This requires some clever geometry in three dimensions, but it might be worth reading through. The write-up at `http://www.math.rutgers.edu/~erowland/polyhedra.html` is a lot slicker than the way I would've gone about things.