18.02 Recitation
Solutions
12 September 2011

Here are writeups for a few of the problems on today's sheet. If you'd like to see the solution to any of the others, send me an email (`johnl@math.mit.edu`).

1. *Let* $\vec{v} = \langle 2, -2, 1\rangle$. *What is the component of* $\vec{v}$ *in the direction of the vector* $\vec{w} = \langle 3, 2, -6\rangle$*?*

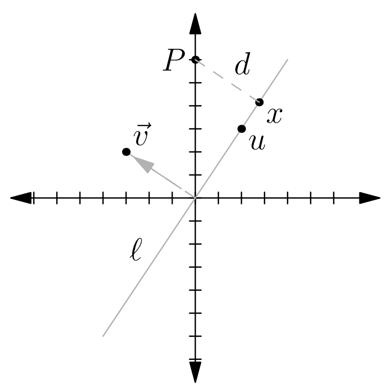
   We use the formula

   $$\text{comp}_{\vec{w}}\, \vec{v} = \vec{v} \cdot \left(\frac{\vec{w}}{\|\vec{w}\|}\right) = \langle 2, -2, 1\rangle \cdot \left(\frac{\langle 3, 2, -6\rangle}{\|\langle 3, 2, -6\rangle\|}\right).$$

   Note that since $\vec{w}$ is not a unit vector, we have to divide through by its magnitude. Since $c\vec{v} \cdot \vec{w} = c\vec{v} \cdot \vec{w}$, we can move the denominator $\|\vec{w}\|$ outside of the dot product:

   $$\begin{aligned}\langle 2, -2, 1\rangle \cdot \left(\frac{\langle 3, 2, -6\rangle}{\|\langle 3, 2, -6\rangle\|}\right) &= \frac{\langle 2, -2, 1\rangle \cdot \langle 3, 2, -6\rangle}{\|\langle 3, 2, -6\rangle\|} \\ &= \frac{(2)(3) + (-2)(2) + (1)(-6)}{\sqrt{3^2 + 2^2 + (-6)^2}} = -\frac{4}{7}.\end{aligned}$$

2. *Let* ℓ *be a line through* $(0, 0)$ *and* $(2, 3)$. *What is the distance from* ℓ *to the point* $(0, 6)$*? Can you find the closest point on the line?*

   Let $\vec{P} = \langle 0, 6\rangle$ be the vector pointing to the point $(0, 6)$, and $\vec{u}$ the vector pointing along the line. pointing in the direction of the line.

   First let's find the closest point. From the picture, we can see that this point x is exactly a distance $\text{comp}_{\vec{u}}\, \vec{P}$ in the direction of $\vec{u}$. That distance is

   $$\text{comp}_{\vec{u}}\, \vec{P} = \frac{\vec{P} \cdot \vec{u}}{\|\vec{u}\|} = \frac{\langle 0, 6\rangle \cdot \langle 2, 3\rangle}{\sqrt{13}} = \frac{18}{\sqrt{13}}.$$

To find the actual point, we just multiply the distance we just computed by a unit vector in the direction of $\vec{u}$.

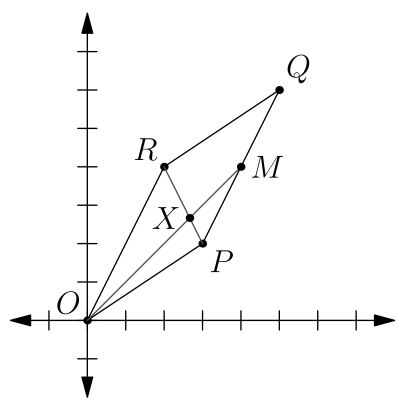
$$\vec{x} = \frac{18}{\sqrt{13}} \cdot \frac{\langle 2,3\rangle}{\sqrt{13}} = \left\langle \frac{36}{13}, \frac{54}{13} \right\rangle.$$

Now we want to find the distance d in the diagram. We could use our computation of $\vec{x}$ and the Pythagorean theorem to find it, but let's use a vector approach. d is the component of $\vec{P}$ in the direction of a vector perpendicular to $\vec{u}$. One such vector is $\vec{v} = \langle -3, 2\rangle$, labelled on the diagram. Then

$$d = \operatorname{comp}_{\vec{v}} \vec{P} = \frac{\vec{P} \cdot \vec{v}}{\|\vec{v}\|} = \frac{\langle 0,6\rangle \cdot \langle -3,2\rangle}{\sqrt{13}} = \frac{12}{\sqrt{13}}.$$

One caveat in my writeup: when doing geometry problems like this, it's sometimes helpful to identify a point (like P) with the vector $\vec{P}$ pointing from the origin to P. Generally this won't cause any ambiguity, but watch out for this point.

3. (1A-12) *Label the four vertices of a parallelogram in counterclockwise order as $OPQR$. Prove that the line segment from O to the midpoint of $\overrightarrow{PQ}$ intersects the diagonal $\overrightarrow{PR}$ in a point X that is $1/3$ of the way from P to R.*



The first thing to do is to write down what the point X is, in terms of the vectors associated to the other points. The vector pointing from P to R is simply $\vec{R} - \vec{P}$. If we start at $\vec{P}$ and go a third of the way along that vector, we end up at

$$\vec{X} = \vec{P} + \frac{1}{3}(\vec{R} - \vec{P}) = \frac{1}{3}\vec{R} + \frac{2}{3}\vec{P}.$$

Is this on the line passing from O to the midpoint of $\overrightarrow{PQ}$? Well, that line passes through $(\vec{P} + \vec{Q})/2$. Since $\vec{Q} = \vec{P} + \vec{R}$, this is

$$\vec{M} = \frac{\vec{P} + \vec{Q}}{2} = \frac{\vec{P} + \vec{P} + \vec{R}}{2} = \vec{P} + \frac{\vec{R}}{2}.$$

Is X on the line through M? The line from O to M consists of all the vectors $c\vec{M}$ where c is any number. From the above calculations we see that

$$\vec{X} = \frac{1}{3}\vec{R} + \frac{2}{3}\vec{P} = \frac{2}{3}\left(\vec{P} + \frac{\vec{R}}{2}\right) = \frac{2}{3}\vec{M},$$

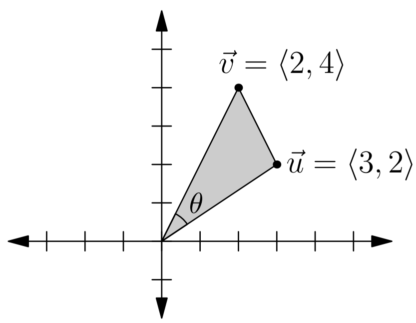
so X is indeed on the line in question.

4. (1C-3). Find the area of the plane triangle whose vertices lie at:

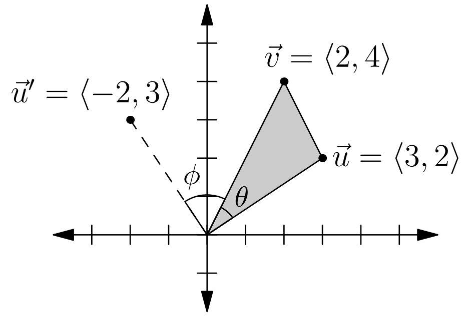
   (a) $(0,0),(1,2),(1,-1)$

   (b) $(1,2),(1,-1),(2,3)$

   There was some confusion about computing the area of a triangle in this way, so I'll go through the calculation again. Suppose we have a triangle with vertices at $(0,0)$, $\vec{u} = \langle u_1, u_2\rangle$, and $\vec{v} = \langle v_1, v_2\rangle$, and let θ be the angle between $\vec{u}$ and $\vec{v}$:



   The area of the triangle is $A = \frac{1}{2}\|\vec{u}\|\|\vec{v}\|\sin\theta$. You can get this from some easy trig, since the area is half the base times the height. Take the base to be along $\vec{u}$: the base is $\|u\|$ and the height is $\|v\|\sin\theta$. Unfortunately this answer isn't $\vec{u}\cdot\vec{v}/2$: then we'd have cosine instead of sine. That means that we don't have any easy way to compute this in terms of the coordinates of the vector (whereas if it were the dot product, we would). The plan is to find another vector $\vec{u}'$ so that the area actually is $\vec{u}'\cdot\vec{v}/2$. To accomplish this, we're going to be a bit sneaky: define a new vector $\vec{u}'$ obtained by rotating $\vec{u}$ 90 degrees counterclockwise. Then $\vec{u}' = \langle -u_2, u_1\rangle$.



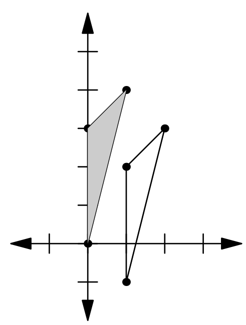
Now, $\phi = 90° - \theta$, so $\sin\theta = \cos\phi$. That means that $A = \frac{1}{2}\|\vec{u}\|\|\vec{v}\|\cos\phi$. But since $\|\vec{u}\| = \|\vec{u}'\|$ (it's just a rotated version of it, after all), in fact

$$A = \frac{1}{2}\|\vec{u}'\|\|\vec{v}\|\cos\phi = \frac{1}{2}\vec{u}' \cdot \vec{v}.$$

That's an actual dot product and we know how to compute it in terms of the coordinates. Since $\vec{u}' = \langle -u_2, u_1\rangle \cdot \langle v_1, v_2\rangle = u_1v_2 - u_1v_2$, the area is $(u_1v_2 - u_2v_1)/2$. If you've studied matrices before, you'll recognize this as $\frac{1}{2}\det\left(\begin{smallmatrix} u_1 & v_1 \\ u_2 & v_2 \end{smallmatrix}\right)$.

Now we apply it to the triangles given. The first is exactly as in our setup. We have one vertex at $(0,0)$, and the others at $\vec{u} = \langle 1, -1\rangle$ and $\vec{v} = \langle 1, 2\rangle$, so that we have $u_1 = 1$, $u_2 = -1$, $v_1 = 1$, and $v_2 = 2$. The formula gives $A = (u_1v_2 - u_1v_2)/2 = ((1)(2) - (1)(-1))/2 = 3/2$.

The second triangle is a bit trickier, since none of the vertices is at the origin. To get around this, we just shift the entire triangle by adding $\langle -1, 1\rangle$ to each of the vertices. This gives us a new triangle that obviously has the same area, but does have a vertex at the origin.



The new triangle has vertices at $\langle 1, 4\rangle$ and $\langle 0, 3\rangle$. Our formula for the area gives $A = (u_1v_2 - u_2v_1) = 3/2$.

5. *(1B-6) Let O be the origin, c a given number, and $\hat{u}$ a given direction (i.e. a unit vector). Describe geometrically the locus of points P in space such that $\overrightarrow{OP} \cdot \hat{u} = c\left|\overrightarrow{OP}\right|$.*

   Since
   $$\overrightarrow{OP} \cdot \hat{u} = \|\overrightarrow{OP}\|\|\hat{u}\|\cos\theta = \|\overrightarrow{OP}\|\cos\theta,$$
   we're asking for all vectors with $\cos\theta = c$, that is, that make an angle of $\cos^{-1}(c)$ with $\hat{u}$. If c isn't between -1 and 1, there are no such vectors (since cosine is always in this range). If it is, we get a cone, with vertex at the origin, pointing in the direction of $\hat{u}$, and with vertex angle $\cos^{-1}(c)$. (If this description isn't clear, think about what happens in a particular case: what are all the vectors that make an angle of 30° with $\hat{\imath} = \langle 1, 0, 0\rangle$?)