18.02 Recitation
Solutions
19 September 2011

1. *($\approx$ 1G-1) Consider the three planes defined by the equations*

$$\begin{aligned} 3x + y - z &= 8 \\ -x + 2y &= 3 \\ -x - y - z &= 0 \end{aligned}$$

Determine the unique point of intersection of the three planes. Hint:

$$\begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}^{-1} = \frac{1}{10}\begin{pmatrix} 2 & -2 & -2 \\ 1 & 4 & -1 \\ -3 & -2 & -7 \end{pmatrix}$$

The given equation is equivalent to the matrix equation

$$\begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$$

To find the $\langle x, y, z\rangle$ satisfying this, we multiply on the left by A^{-1}:

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} 2 & -2 & -2 \\ 1 & 4 & -1 \\ -3 & -2 & -7 \end{pmatrix}\begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \end{aligned}$$

So the unique solution to the system is $\langle 1, 2, -3\rangle$.

2. *The first two planes above intersect in a line. Give a parametric equation for the line. Check your answer to the previous question by finding the intersection of your parametrized line with the plane.*

I messed this one up in recitation, so here's a complete solution.

To specify the line that is the intersection of the two planes, we're going to need to do two things: find a direction vector (call it $\vec{v}$) of the line, and find one point (specified by the vector $\vec{p}$) from $(0, 0, 0)$ to the point) on the line. Then we can write down the line in parameterized form as $\vec{p} + \vec{v}t$.

So the first order of business is to find the direction. Well, the line is contained in both of the planes, since it's in their intersection. That means that it's perpendicular to the normal vector of each plane. From the equations we can read off these normal vectors:

$$\begin{aligned} \vec{n}_1 &= \langle 3, 1 - 1\rangle \\ \vec{n}_2 &= \langle -1, 2, 0\rangle. \end{aligned}$$

The direction $\vec{v}$ is normal to both of these. And we know how to find a vector normal to both of two given vectors: it's just the cross product of the two (in fact, any other vector that's perpendicular to both must be a multiple of the cross product).

Therefore we compute

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 3, 1-1 \rangle \times \langle -1, 2, 0 \rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -1 \\ -1 & 2 & 0 \end{vmatrix}$$
$$= (0-(-2))\hat{\imath} + (1-0)\hat{\jmath} + (6-(-1))\hat{k} = \langle 2, 1, 7 \rangle.$$

This is the direction of the line. Why take $\vec{n}_1 \times \vec{n}_2$ instead of $\vec{n}_2 \times \vec{n}_1$? It doesn't actually make any difference for our purposes – we'd parametrize the same line, only our parametrization would move in the opposite direction.

The other thing to do is find a point on the line, i.e. one solution to the system of two equations. This is a lot easier than find all of them. If we can find x and y with $3x + y = 8$ and $-x + 2y = 3$ we can just take $z = 0$ and we're set. We find that $x = 13/7$, $y = 17/7$ does the trick, and so $\vec{p} = (13/7, 17/7, 0)$ is on the line. So the intersection is given by

$$\ell(t) = \left\langle \frac{13}{7}, \frac{17}{7}, 0 \right\rangle + \langle 2, 1, 7 \rangle t.$$

To check our answer to the previous question, we can find the intersection of this line with the plane. To do that, we just need to find the value of t for which $\langle 13/7 + 2t, 17/7 + t, 7t \rangle$ satisfies $-x - y - z = 0$:

$$-\left(\frac{13}{7} + 2t\right) - \left(\frac{17}{7} + t\right) - 7t = 0$$
$$-\frac{30}{7} - 10t = 0$$
$$t = -\frac{3}{7}$$

The intersection is the point that corresponds to this value of the parameter:

$$\ell(-10/7) = \ell(t) = \left\langle \frac{13}{7}, \frac{17}{7}, 0 \right\rangle + \langle 2, 1, 7 \rangle \left(-\frac{3}{7}\right) = \langle 1, 2, -3 \rangle.$$

This is the same answer we got before!

3. *Consider the two lines given by the parametric equations* $\vec{a}(s) = \langle 1, 0, 0 \rangle + \langle 1, 1, 1 \rangle s$ *and* $\vec{b}(t) = \langle 0, 0, 0 \rangle + \langle 0, 0, 1 \rangle t$. *Do they intersect? What is the shortest distance between the two?*

The lines intersect if we can find values of s and t such that $\vec{a}(s) = \vec{b}(t)$. Note that we don't want $\vec{a}(s) = \vec{b}(s)$: the intersection doesn't have to occur for the same value of the parameter on both lines!

This happens if there are s and t such that the two agree in all three coordinates:

$$1 + 1 \cdot s = 0 + 0 \cdot t, \quad 0 + 1 \cdot s = 0 + 0 \cdot t, \quad 0 + 1 \cdot s = 0 + 1 \cdot t$$

The first equation forces $1 + s = 0$, so $s - 1$. But then the second can't possibly be satisfied. So there is no way these lines intersect.

Finding the distance between them is harder. The first observation is that if you draw a segment between the two closest points, it will be perpendicular to both of the lines:

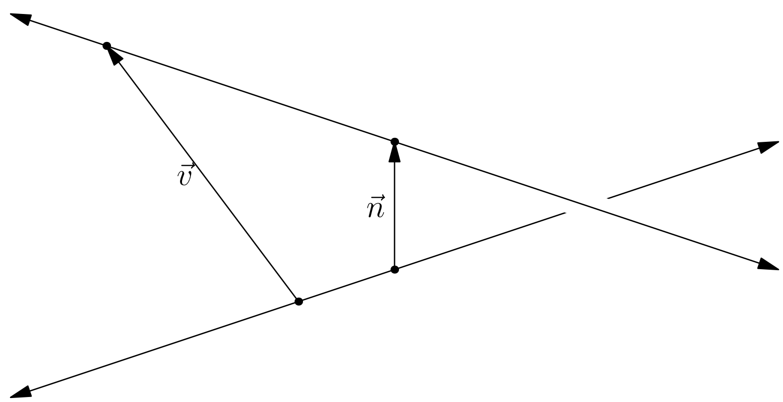


Figure 1: $d = \text{comp}_{\vec{n}}\, \vec{v}$

Thus the direction of this segment must be given by the cross product of the direction vectors for the two lines!

$$\vec{n} = \langle 1, 1, 1 \rangle \times \langle 0, 0, 1 \rangle = \langle 1, -1, 0 \rangle$$

Now, imagine drawing a vector between any point on one of the lines and any point on the other; this is $\vec{v}$ in the picture. Its component in the direction $\vec{n}$ is precisely the distance between the two lines. In doesn't matter what two points you pick.

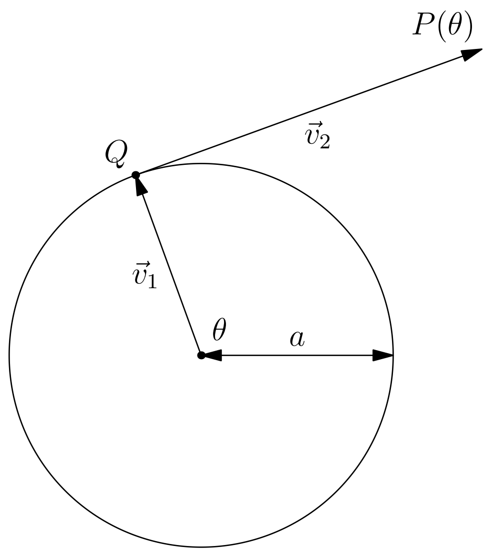
Maybe you can see this from the picture. If not, try to convince yourself that this is the case when one of the lines is the x-axis, and the other is parallel to the y-axis but shifted one unit in the z direction. We can find the distance between the lines (which is obviously 1) by taking the vertical part of any old segment connecting one point on each line.

In our case, there's an obvious vector connecting the two lines: just use the one between the two points in the parametrization. This is $\langle 1, 0, 0 \rangle - \langle 0, 0, 0 \rangle = \langle 1, 0, 0 \rangle$. So the distance is given by

$$\text{comp}_{\vec{n}} \langle 1, 0, 0 \rangle = \frac{\langle 1, 0, 0 \rangle \cdot \langle 1, -1, 0 \rangle}{\| 1, -1, 0 \|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

There are various other ways to think about finding the distance between two lines. If you don't like this one, Google around a bit.

4. *(1I-4) A roll of plastic tape of outer radius a is held in a fixed position while the tape is being unwound counterclockwise. The end P of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin O, and the end P to be initially at $(a, 0)$, write parametric equations for the motion of P.*



Let's figure out where we are when we've unwound an angle θ. Our position at this time is going to be given as the sum of two parts: one a radial vector $\vec{v}_1$ pointing from the center of the tape to the point where the unwound tape meets the spool (Q in the picture), and another from this point, in a direction 90 degrees clockwise, of length equal to the amount of tape we've unwound. That's $\vec{v}_2$.

We have $Q = (a\cos\theta, a\sin\theta)$. Rotated 90 degrees clockwise, this is the unit vector $\langle \sin\theta, -\cos\theta \rangle$. That length of tape unwound when we've gone around θ is $2\pi a\theta$ by the formula for length of an arc on the circle. So $\vec{v}_2 = 2\pi a\theta \langle \sin\theta, -\cos\theta \rangle$.

So the position $P(\theta)$ is just given by

$$\langle a\cos\theta, a\sin\theta \rangle + 2\pi a\theta \langle \sin\theta, -\cos\theta \rangle = \langle a\cos\theta + 2\pi a\theta \sin\theta, a\sin\theta - 2\pi a\theta\cos\theta \rangle.$$

5. *Find the equation of the plane passing through* $(1, 0, 0)$, $(2, 3, -2)$, *and* $(1, 1, 1)$. To find the equation of the plane, it's good enough to find the normal vector and a point through which it passes. The normal vector is perpendicular to any vector contained in the plane, and so we can compute it as the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, 3, -2 \rangle \times \langle 0, 1, 1 \rangle = \langle 5, -1, 1 \rangle.$$

A plane normal to $\langle 5, -1, 1 \rangle$ and through $(1, 0, 0)$ is given by

$$5(x - 1) - y + z = 0.$$

A quick check shows that all three points do indeed satisfy this equation.

6. *How would you find the equation of a plane containing a given line and passing through a given point? Contain two given points and orthogonal to a given plane?*

   There are lots of problems in the genre of finding lines and planes passing through or perpendicular to various other lines and planes. These are just a couple examples. The general strategy for these sorts of things is to try to find the relations that must be satisfied between normal vectors to planes, directions vectors of lines, and vectors between points. I know this is pretty vague advice, but the strategy will depend on exactly what the setup is.

   To find a plane containing a given line and passing through some point, you can pick two random points Q and R on the line. The plane must be normal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and so the normal vector is given by the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$. The plane goes through the point P, so this is enough to write down an equation.

   The condition that two planes be orthogonal is equivalent to asking that the normal vectors $\vec{n}_1$ and $\vec{n}_2$ are orthogonal (try to draw a picture and convince yourself of this). We want the normal vector of our plane to be orthogonal to the normal vector of the given plane (let's call the latter $\vec{n}$). Our normal vector should also be orthogonal to the direction vector $\vec{v}$ of the line, since the line is contained in the plane. Thus our normal vector is given by $\vec{n} \times \vec{v}$. Once you've figured this out, you can write down the equation of a plane normal to $\vec{n} \times \vec{v}$ and passing through one of the points given.