18.02 Recitation
Problems
05 October 2011

1. *Find the best fit line through the three points* $(-1,0)$, $(0,1)$, *and* $(2,2)$.

   We found that the line is given by $y = ax + b$, where a and b satisfy

$$\left(\sum_{i=1}^{3} x_i^2\right) a + \left(\sum_{i=1}^{3} x_i^2\right) b = \sum_{i=1}^{3} x_i y_i$$
$$\left(\sum_{i=1}^{3} x_i\right) a + nb = \sum_{i=1}^{3} y_i$$

   Plugging in the points in question, we get $5a + b = 4$, $a + 3b = 3$. Solving we get $a = 9/14$, $b = 11/14$. If you sketch the points and the line, you'll see that this is a pretty good match.

2. *How would you fit a quadratic function* $y = ax^2 + bx + c$ *to through a collection of points* (x_i, y_i)*?*

   Our error function this time would be

$$E(a,b,c) = \sum_{i=1}^{n} \left(y_i - (ax_i^2 + bx_i + c)\right)^2 .$$

   This time set the three partials $\frac{\partial E}{\partial a}$, $\frac{\partial E}{\partial b}$, $\frac{\partial E}{\partial c}$ all zero and solve for a, b, c as before.

3. Compute the gradient of the function $f(x,y) = \sqrt{xy}$. What is its derivative in the direction $\langle 1,1\rangle$ at the point $(x,y) = (1,2)$?

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle .$$

   At $(x,y) = (1,2)$, this is $\left\langle \frac{2}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right\rangle$.

   The direction vector $\hat{u}$ is $\langle \sqrt{2}/2, \sqrt{2}/2 \rangle$. Then

$$D_{\hat{u}} f = \hat{u} \cdot \nabla f = \frac{2}{2\sqrt{2}} \frac{\sqrt{2}}{2} + \frac{1}{2\sqrt{2}} \frac{\sqrt{2}}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

4. *(2E-2d) Suppose* $f(x,y,z)$ *is a function with* $\nabla f = \langle 3x^2 y, x^3 + z, y\rangle$. *Let* $x = t$, $y = t^2$, *and* $z = t^3$. *Find* $\frac{df}{dt}$.

   The chain rule says that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

We get the first factor in each term out of the gradient, and the second by computing the given derivatives.

$$\frac{df}{dt} = (3x^2y)(1) + (x^3+z)(2t) + (y)(3t^2) = 3t^4 + (t^3+t^3)(2t) + (t^2)(3t^2) = 10t^4.$$

5. *(13.7.38) Three resistors of resistance $R_1 = 100\,\Omega$, $R_2 = 100\,\Omega$, $R_3 = 200\,\Omega$ are wired in parallel, with total resistance*

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Suppose R_1 and R_2 increase at the rate of $1\,\Omega/s$, while R_3 decreases at $2\,\Omega/s$. At what rate (and in what direction) does R change?

We know that

$$\frac{dR}{dt} = \frac{\partial R}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial R}{\partial R_2}\frac{dR_2}{dt} + \frac{\partial R}{\partial R_3}\frac{dR_3}{dt}$$

To find $\frac{\partial R}{\partial R_i}$, we use implicit differentiation, which gives

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}.$$

So $\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}$, etc. The derivatives $\frac{dR_i}{dt}$ are given in the problem statement. We're all ready to plug everything in, except we need to know the initial value of R. Using the defining formula we have $1/R = 1/100 + 1/100 + 1/200$, whence $R = 40\,\Omega$. So

$$\frac{dR}{dt} = \frac{40^2}{100^2}(1) + \frac{40^2}{100^2}(1) + \frac{40^2}{200^2}(-2) = -\frac{2}{25}.$$

So R is decreasing at a rate of $0.04\,\Omega/s$.

6. *A cylinder of radius $r = 5$ and height $h = 5$ expands, with the radius growing at a rate of 1 ft/sec, and the height at 2 ft/sec. Compute dV/dt.*

Well, $V = \pi r^2 h$. The chain rule gives

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = (2\pi r h)\frac{dr}{dt} + (\pi r^2)\frac{dh}{dt} \\ &= (50\pi)(1) + (25\pi)(2) = 100\pi \text{ ft}^3/\text{sec}\end{aligned}$$