18.02 Recitation
Problems
17 October 2011

1. (2J-8) *Take (r,θ) as polar coordinates. Let $w=\sqrt{r^2-x^2}$. Compute $\left(\frac{\partial w}{\partial r}\right)_\theta$ using our three different methods.* The easiest way is to write w as a function of r and θ alone, eliminating x using the constraint $x = r\cos\theta$. In this method, we get

$$w=\sqrt{r^2-x^2}=\sqrt{r^2-r^2\sin^2\theta}=r\sin\theta,$$

whence $\left(\frac{\partial w}{\partial r}\right)_\theta = \sin\theta$.

We can also use the chain rule:

$$\begin{aligned}\left(\frac{\partial w}{\partial r}\right)_\theta &= \left(\frac{\partial w}{\partial x}\right)_\theta \left(\frac{\partial x}{\partial r}\right)_\theta + w_r \\ &= -\frac{x}{\sqrt{r^2-x^2}}\left(\frac{\partial x}{\partial r}\right)_\theta + \frac{r}{\sqrt{r^2-x^2}}.\end{aligned}$$

Since $x = r\cos\theta$, $\left(\frac{\partial x}{\partial r}\right)_\theta = \sin\theta$. So our answer is

$$\left(\frac{\partial w}{\partial r}\right)_\theta = -\frac{x}{\sqrt{r^2-x^2}}\cos\theta + \frac{r}{\sqrt{r^2-x^2}}.$$

Using differentials,

$$dw = \frac{r}{\sqrt{r^2-x^2}}\,dr - \frac{x}{\sqrt{r^2-x^2}}\,dx.$$

But $r\cos\theta = x$, so $dx = \cos\theta dr + r\sin\theta\, d\theta$. Thus

$$dw = \left(\frac{r}{\sqrt{r^2-x^2}} - \frac{x}{\sqrt{r^2-x^2}}\cos\theta\right)dr - \frac{x}{\sqrt{r^2-x^2}} r\sin\theta\, d\theta$$

So

$$\left(\frac{\partial w}{\partial r}\right)_\theta = -\frac{x}{\sqrt{r^2-x^2}}\cos\theta + \frac{r}{\sqrt{r^2-x^2}}.$$

To check that the latter two answers agree with the first, observe that

$$\begin{aligned}-\frac{x}{\sqrt{r^2-x^2}}\cos\theta + \frac{r}{\sqrt{r^2-x^2}} &= \frac{r}{r\sin\theta} - \frac{r\cos\theta}{r\sin\theta}\cos\theta \\ &= \frac{1-\cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta.\end{aligned}$$

2. (2J-10) *Suppose $u(x,y)$ and $v(x,y)$ are functions of two variables. Use differentials to prove the Jacobian rule*

$$\left(\frac{\partial u}{\partial x}\right)_v = \left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial u}{\partial v}\right)_x \left(\frac{\partial v}{\partial x}\right)_y.$$

The first step is to compute $\left(\frac{\partial u}{\partial x}\right)_v$. Well,

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy.$$

From the constraint we have $0 = dv = \frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy$. Thus on the constrained surface $dy = -(\partial v/\partial x)/(\partial v/\partial y)\, dx$. Plugging this back in,

$$du = \left(\frac{\partial u}{\partial x} - \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \frac{\partial v}{\partial x} \right) dx$$

On the other hand, $\left(\frac{\partial u}{\partial x}\right)_y$ is just plain old $\frac{\partial u}{\partial x}$, since this is unconstrained (u is a function of x and y in the first place). Similarly $\left(\frac{\partial v}{\partial x}\right)_y = \frac{\partial v}{\partial x}$. We just need to compute $\left(\frac{\partial u}{\partial v}\right)_x$, using the constraint. With x fixed, $du = \frac{\partial u}{\partial y}\, dy$ (the dx term is 0), and $dv = \frac{\partial v}{\partial y}\, dy$. So $\left(\frac{\partial u}{\partial v}\right)_x = \frac{\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}}$. Plugging all these computations back into the original thing, you see that the claimed equality holds.

I'll postpone all these integration problems to Wednesday's set – see those solutions.