Contour lines/surfaces: (2A-1, 09/28 recitation problems)

- For a function $f(x, y)$ (resp. $f(x, y, z)$), these are defined by $f(x, y) = 0$ (resp. $f(x, y, z) = 0$)
- Perpendicular to gradient of f
- Be able to estimate the gradient and directional derivatives from a contour plot (e.g. topographic map)
- Tangent to contour line is a direction with zero directional derivative

Partial derivatives: (2A-2, 2B-5, 2B-9)

- $\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ = "What is the instantaneous rate of change of f when we change x?"
- Compute by treating y as a constant and pretending f is a function of the single variable x.
- Linear approximation:

$$f(x + \Delta x, y + \Delta y, z + \Delta z) \approx f(x, y, z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$$

Gradient: (2B-1, 2D-1)

- Defined by $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$
- It's a vector (which depends on x, y, and z)
- The tangent plane to a level surface $g(x, y, z)$ is perpendicular to the gradient: you can use this to find tangent planes

Directional derivative: (2D-2, 2D-4)

- For a direction vector $\hat{u}$, defined as the limit

$$\left.\frac{df}{ds}\right|_{\hat{u}} = \lim_{h\to 0} \frac{f(\vec{x} + h\vec{u})}{h}$$

- "What is the instantaneous rate of change of f when we change $\vec{x}$ in the direction of $\hat{u}$?"
- Compute it using $\left.\frac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}$.
- That's equal to $|\nabla f| \cos\theta$, where θ is the angle between ∇f and $\hat{u}$. So directional derivative is 0 if $\hat{u}$ is perpendicular to ∇f, maximum if $\hat{u}$ is in the direction of ∇f, and minimum if $\hat{u}$ is in the opposite direction.

Chain rule: (2E-1, 2E-5)

- If $f(u, v)$ is a function of u and v, with $u(x, y)$ and $v(x, y)$ are functions of x and y, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

  Same idea for other dependences – see the book.

Max-min problems: (2F-1, 2F-5)

- Maxima and minima occur at critical points, where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, or at the boundary.
- First step is to set up the problem: figure out the set of x and y over which you want to maximize.
- Find the critical points, check values there, and find max/min on the boundary (e.g. by parametrizing)

Least squares: (2G-1)

- This is an application of our max-min techniques.
- Given a bunch of data points, (x_i, y_i) (for $1 \leq i \leq n$), want to find a line (or other function) that approximates them as well as possible.
- The idea: think of the error $\sum_{i=1}^{n}(y - (ax_i + b))^2$ as a function of the parameters a and b, and then find the a and b for which this is minimized.

Second derivative test: (2H-1, 2H-5)

- Used to check if a critical point is a maximum or a minimum
- Compute
$$A = \frac{\partial^2 f}{\partial x^2}, \quad B = \frac{\partial^2 f}{\partial x\, \partial y}, \quad C = \frac{\partial^2 f}{\partial y^2}$$
- $AC - B^2 > 0, A > 0 \implies$ min, $AC - B^2 > 0, A < 0 \implies$ max, $AC - B^2 < 0 \implies$ saddle point, $AC - B^2 = 0 \implies$ can't say anything.

Lagrange multipliers: (2I-1, 2I-3)

- Want to maximize $f(x, y, z)$ subject to the condition $g(x, y, z) = c$.
- This will happen at a point where $\nabla f = \lambda \nabla g$ for some constant λ (i.e. where the two gradients are parallel to each other).

Non-independent variables: (2J-2, 2J-6)

- Derivative with constraints: subject to $g(x, y, z) = c$, how does f change when x changes?
- The constraint means that when we change x, y and/or z much change too in order to ensure that g stays constant.
- $\left(\frac{\partial f}{\partial x}\right)_z$ is the derivative we get when changing x, holding z constant, and changing y in order to keep $g(x, y, z) = c$.
- Find change in y in terms of change in x, while having 0 change in z: one way is to use $0 = dg = g_x\, dx + g_y\, dy + g_z\, dz$.
- Three ways to compute it: see Lecture Notes 14.

PDE

- A PDE is a differential equation involving the partial derivatives of a function, i.e. something like $f_{xx} + f_{yy} + f_{zz} = 0$.
- You should understand what such an equation means and be able to verify that a given function is a solution.