18.02 Recitation
Problems
2 November 2011

1. *Integrate the field* $\vec{F}(x,y) = \hat{\imath}$ *over a circular path from* $(1,0)$ *to* $(0,1)$*. First do this by parametrizing the path, and then use the fact that it is a gradient field and apply the fundamental theorem of calculus for line integrals.*

   We parametrize the path by $\vec{r}(t) = \langle \cos t, \sin t\rangle$. This has velocity $d\vec{r}/dt = \langle -\sin t, \cos t\rangle$, and so

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{\pi/2} \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}\,dt = \int_{t=0}^{\pi/2} \langle 1,0\rangle\cdot\langle -\sin t, \cos t\rangle\ dt$$
$$= \int_{t=0}^{\pi/2} -\sin t\,dt = -1.$$

   To do this with the fundamental theorem, observe that the field $\vec{F}$ is the gradient of $f(x,y) = x$. So $\int_C \vec{F}\cdot d\vec{r} = f(0,1) - f(1,0) = 0 - 1 = -1$.

2. *Compute*

$$\int_C (4x^3 + 6xy^2)\,dx + (6x^2y + 8y^3)\,dy$$

   where C *is a straight line from* $(0,1)$ *to* $(1,0)$ *using the definition of a line integral.*

   The line is parametrized using $\vec{r}(t) = \langle t, 1-t\rangle$. This has $d\vec{r}/dt = \langle 1,-1\rangle$. Plugging in our formula for $\vec{r}$ to $\vec{F}$, we have $\vec{F}(\vec{r}(t)) = \langle 4t^3 + 6t(1-t)^2, 6t^2(1-t) + 8(1-t)^3\rangle$. Then the definition for a line integral gives

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} \left\langle 4t^3 + 6t(1-t)^2, 6t^2(1-t) + 8(1-t)^3\right\rangle \cdot 1, -1\,dt$$
$$= \int_{t=0}^{1} \big((4t^3 + 6t(1-t)^2)(1) + (6t^2(1-t) + 8(1-t)^3)(-1)\big)\ dt = -1.$$

3. *Two of the following four fields are gradient fields. Figure out which are, and for each, give the corresponding potential function.*

   (a) $\vec{F} = \langle 4x^3 + 6xy^2, 6x^2y + 8y^3\rangle$

   (b) $\vec{F} = \langle x^2 \sin y, -xy\sin y\rangle$

   (c) $\vec{F} = \langle ye^y - y\sin(xy), xe^y - x\sin(xy)\rangle$

   (d) $\vec{F} = \langle 6xy^3 + 1, 9x^2y^2\rangle$

   In each case, we need to check whether $M_y = N_x$:

(a)

$$M_y = 12xy,$$
$$N_x = 12xy.$$

So this is conservative. If it's the gradient of a function f, we must have $f_x = M = 4x^3 + 6xy^2$. This means that $f = x^4 + 3x^2y^2 + g(y)$, for some function g that depends only on y. To figure out what g, we use the fact that $f_y = N$. $f_y = 6x^2y + g'(y)$, which agrees with N if $g'(y) = 8y^3$. This means $g(y) = 2y^4$. Putting this all together we get $f(x, y) = x^4 + 3x^2y^2 + 2y^4$.

(b)

$$M_y = x^2 \cos y,$$
$$N_x = -y \sin y.$$

$M_y \neq N_x$, so this field is definitely not conservative.

(c)

$$M_y = e^y + ye^y - xy\cos(xy) - \sin(xy),$$
$$N_x = e^y - xy\cos(xy) - \sin(xy).$$

Not equal, so this field isn't conservative either.

(d)

$$M_y = 18xy^2$$
$$N_x = 18xy^2$$

These are equal. We find the potential function as before. $f_x = M = 6xy^3 + 1$, so $f = 3x^2y^3 + x + g(y)$. Then $f_y = 9x^2y^2 + g'(y)$, and in this case $g'(y) = 0$, so $g(y)$ is a constant c, and our function is $f(x, y) = 9x^2y^2 + c$.

4. *Double-check your answer to problem 2 by using the fundamental theorem.*

The field in problem two turned out to be the gradient of $x^4 + 3x^2y^2 + 2y^4$. The fundamental theorem tells us that the integral in that question is equal to $f(1, 0) - f(0, 1) = 1 - 2 = -1$, which agrees with our other computation.

5. *What are the maximum and minimum values that can be achieved by an integral*

$$\int_C (4x^3 + 6xy^2)\,dx + (6x^2y + 8y^3)\,dy,$$

where C is a path contained inside the unit circle?

The value of the integral is $f(x_2, y_2) - f(x_1, y_1)$, where C starts at (x_1, y_1) and ends at (x_2, y_2). To make this as big as possible, we want to find (x_2, y_2) such that $f(x_2, y_2)$ is as large as possible, and (x_1, y_1) such that $f(x_1, y_1)$ is as small as possible. Then the maximum value of the integral will be obtained for any path between these two points.

Actually finding the values is a standard max/min problem. If you work through it using the methods from last unit, you'll find that the min is at $(0,0)$, with $f(0,0)=0$, and the max is at $(0,1)$, with a value 2. So the largest the integral can possibly be is $2-0=2$.

6. *Find streamlines of the vector field* $\vec{F}(x,y)=\langle -y,x\rangle$.

   We want to know what paths $\vec{r}(t)$ are streamlines of the field (basically solving for $\vec{r}$, given $\vec{F}$). For a function to be a streamline we should have $d\vec{r}/dt=\vec{F}(\vec{r}(t))$. If $\vec{r}(t)=\langle x(t),y(t)\rangle$, then $\vec{F}(\vec{r}(t))=\langle -y(t),x(t)\rangle$, and $d\vec{r}/dt=\langle x(t),y(t)\rangle$. This means that $dx/dt=-y(t)$ and $dy/dt=x(t)$. This is a system of two PDEs, and we need to find $x(t)$ and $y(t)$ to satisfy both simultaneously. In general this is a pretty hard thing to do. In this case, we can differentiate the first equation with respect to x, getting $d^2x/dt^2=d/dt(-y(t))=-x(t)$. This is solved by $x(t)=a\cos t+b\sin tc$. Then $y(t)=-dx/dt=a\sin t-b\cos t$. This are the streamlines.