18.02 Recitation
Problems
7 November 2011

1. *(4F1) Compute* $\operatorname{div} F$ *and* $\operatorname{curl} F$ *for the fields* $\vec{F}_1(x,y) = a\,\hat{\imath} + b\hat{\jmath}$, $\vec{F}_2(x,y) = x^2\,\hat{\imath} + y^2\,\hat{\jmath}$, $\vec{F}_3(x,y) = xy(\hat{\imath} + \hat{\jmath})$.

   To get these, just take the formulas defining curl and divergence and plug in the equations for the vector fields. You obtain $\operatorname{div} \vec{F}_1 = 0$, $\operatorname{curl} \vec{F}_1 = 0$, $\operatorname{div} \vec{F}_2 = 2x + 2y$, $\operatorname{curl} \vec{F}_2 = 0$, $\operatorname{div} \vec{F}_3 = x + y$, $\operatorname{curl} \vec{F}_3 = y - x$.

2. *(4D1a) Let* $\vec{F}(x,y)$ *be the vector field* $2y\,\hat{\imath} + x\hat{\jmath}$ *and* C *the unit circle (oriented counter clockwise). Evaluate* $\int_C \vec{F}\,d\vec{r}$ *both directly, using the definition of a line integral, and using Green's theorem to find an equivalent double integral.*

   First let's compute directly. Parametrize the path by $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle \cos t, \sin t\rangle$. Then plug in this x and y to the line integral we want to compute.

$$\oint_C 2y\,dx + x\,dy = \int_0^{2\pi} (-2\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} (1 - 3\sin^2 t)\,dt = -\pi.$$

   On the other hand, Green's theorem tells us that $\oint CM\,dx + N\,dy = \iint N_x - M_y\,dA$. We use this formula with $M = 2y$, $N = x$, and get

$$\iint_R (N_x - M_y)\,dA = \iint_R (1-2)\,dA = -\pi,$$

   since we are integrating the function -1 over a circle, which has area π.

3. *Use Green's theorem to compute* $\oint_C (y^2 + y)dx - 2xy\,dy$ *where* C *follows the curve* $y = 1 - x^2$ *from* $(1,0)$ *to* $(-1,0)$ *and then the* x*-axis, from* $(-1,0)$ *back to* $(1,0)$.

   The idea is similar here. First let's compute it as a line integral. We need to break C into two paths, C_1 and C_2. C_1 is the curved part, and C_2 the bit along the x-axis. To parametrize C_1, we can take $x(t) = -t$, $y(t) = 1 - (-t)^2 = 1 - t^2$, and let t run from -1 to 1. For C_2, just take $x(t) = t$, $y(t) = 0$, where t goes from -1 to 1. Then plug in these x and y to the integral, and you get

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}\\ &= \int_{-1}^{1} \left((1-t^2)^2 + (1-t^2)\right)(-dt) - 2(-t)(1-t^2)(-2t\,dt)\\ &\qquad + \int_{-1}^{1} (0^2+0)(1\,dt) - 2t(0)(0\,dt)\\ &= \int_{-1}^{1} 3t^4 - t^2 - 2\,dt + 0 = -\frac{52}{15}.\end{aligned}$$

We can also do it using Green's theorem. Here M is $y^2 + y$ and N is $-2xy$. Then $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (-2y - 2y - 1)\, dA = \iint_R (-4y - 1)\, dA$. The bounds for the parabolic region have x from -1 to 1 and y from 0 to $1 - x^2$. So we set up the integral as

$$\begin{aligned}\iint_R (-4y-1)\, dA &= \int_{x=-1}^{1} \int_{y=0}^{1-x^2} (-4y-1)\, dy\, dx \\ &= \int_{x=-1}^{1} \left(-2y^2 - y\right)\Big|_0^{1-x^2} dx \\ &= \int_{x=-1}^{1} \left(-2(1-x^2)^2 - (1-x^2)\right) dx = -\frac{52}{15}.\end{aligned}$$

4. *Find the area of the ellipse $x^2/a^2 + y^2/^2 = 1$, using the identity*

$$\oint x\, dy = \iint_R 1\, dA.$$

Find the area of a region is one of the most important applications of Green's theorem. We parametrize the boundary via $\langle x(t), y(t)\rangle = \langle a\cos t, b\sin t\rangle$. Green's theorem then tells us that the area is

$$\begin{aligned}A = \oint_C x\, dy &= \int_{t=0}^{2\pi} (a\cos t)(b\cos t\, dt) = \int_{t=0}^{2\pi} ab\cos^2 t\, dt \\ &= ab\int_{t=0}^{2\pi} \frac{1+\cos 2t}{2}\, dt = ab\left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = \pi ab.\end{aligned}$$

This is definitely right when $a = b = 1$ and we have a circle, so it seems pretty plausible.

5. *Consider the vector field $\vec{F}(x,y) = \hat{\imath}$ on the unit circle. Parametrize the circle by $\vec{r}(t) = (\cos t, \sin t)$. For which values of t is $\vec{F} \cdot \vec{n}$ positive? For which is it negative? Evaluate $\int_C \vec{F} \cdot \vec{n}\, ds$ using Green's theorem, and explain why the answer makes sense.*

   It's positive at points where a rightward pointing vector field points out the the circle. This is exactly for points on the right half of the circle: sketch the normal vector as you go around the circle to see this geometrically. Green's theorem tells us that

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA = \iint_R 0\, dA = 0.$$

   The reason this makes sense is that the left half of the circle has negative flux, and it will exactly balance the positive flux on the right half.

6. *(4F3) Verify Green's theorem in the normal form by calculating both sides and showing they are equal if $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$, and C is formed by the upper half of the unit circle and the x-axis interval* $[1, 1]$.

   I'll leave this one to the solutions in the course notes.