18.02 Recitation
Problems
23 November 2011

1. Let $\vec{F}$ be the field $\vec{F}(x,y,z) = \langle x, y, z\rangle$. What is the flux of $\vec{F}$ across the unit sphere? Across a sphere of radius a? Try to find these without doing any computation.

   On the unit sphere, $|\vec{F}| = 1$, so $\vec{F}\cdot\hat{n} = 1$ since the field is parallel to the normal vector, which also has length 1. This means that $\iint \vec{F}\cdot\hat{n} = A = 4\pi$. A similar things work for other spheres. On a sphere of radius a, $|\vec{F}| = a$, and so $\vec{F}\cdot\hat{n} = a$. The integral is then $(4\pi a^2)(a) = 4\pi a^3$.

2. What are some vector fields that have flux 0 across the unit sphere? What are same that are tangent to the unit sphere at every point?

   A constant field pointing in any direction is one example. Anything with divergence zero, in fact. More generally any field whose divergence is 0 is going to work, by the divergence theorem.

   You might try to come up with an example where the field is tangent to the sphere at every point, like the example of the field $\langle y, -x\rangle$ in two dimensions. In fact there's a theorem (the hairy ball theorem) that says that any field that is tangent to the sphere has to be equal to 0 at at least two points.

3. Consider the field $\vec{F}$ from the first problem. Compute the flux across a square in space with vertices at $(\pm 1, \pm 1, 1)$ by integrating directly.

   For this one $d\vec{S}$ is just $\hat{k}\,dx\,dy$. So

$$\int_{x=-1}^{1}\int_{y=-1}^{1} \langle x, y, 1\rangle \cdot 0, 0, 1\,dx\,dy = 4.$$

4. Find the flux of $\vec{F}$ across a cylinder whose base is a in the xy-plane of radius 2 and whole height is 3. Include both the side of the cylinder and its top and bottom.

   We want to get both the sides of the cylinder and the top and bottom bits. We're going to need to integrate over all of them separately. For the sides of the cylinder, use $\hat{n} = \frac{1}{2}\,dz\,d\theta$. Also $dS = a\,dz\,d\theta$. So the integral is

$$\int_{z=0}^{3}\int_{\theta=0}^{2\pi} \langle 2\cos\theta, 2\sin\theta, z\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle\,(2d\theta\,dz) = \int_{z=0}^{3}\int_{\theta=0}^{2\pi} 4\,d\theta\,dz = 24\pi.$$

   We also need to integrate over the top and bottom of the cylinder. The top has $\hat{n} = \hat{k}$, and

$$\int_{r=0}^{2}\int_{\theta=0}^{2\pi} \langle r\cos\theta, r\sin\theta, 3\rangle \cdot 0, 0, 1 = 12\pi$$

   The bottom has $\hat{n} = -\hat{k}$, with

$$\int_{r=0}^{2}\int_{\theta=0}^{2\pi} \langle r\cos\theta, r\sin\theta, 0\rangle \cdot \langle 0, 0, -1\rangle = 0.$$

So the total integral is 36π.

5. Find the flux of $\vec{F}$ across the unit sphere. This time do it by computing directly.

   Use McKernan's formula. $\hat{n} = \frac{1}{a}\langle x, y, z\rangle$ and $dS = \hat{n}a^2 \sin\phi d\phi\, d\theta$. So we compute

$$\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin\phi \langle x, y, z\rangle \cdot \langle x, y, z\rangle \, d\theta\, d\phi = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin\phi\, d\theta\, d\phi = 2\pi(2) = 4\pi.$$

   This agrees with the answer we get from the divergence theorem.

6. Find the flux of $\vec{F}$ across the paraboloid $z = x^2 + y^2$ in the region over the square with vertices $(\pm 1, \pm 1)$. Interpret the sign of your answer.

   Here $d\vec{S} = \langle -2x, -2y, 1\rangle \, dx\, dy$ from the formula. So

$$\begin{aligned} flux &= \int_{x=-1}^{1}\int_{y=-1}^{1} \left\langle x, y, x^2 + y^2\right\rangle \cdot \langle -2x, -2y, 1\rangle \, dx\, dy \\ &= \int_{x=-1}^{1}\int_{y=-1}^{1} -2x^2 - 2y^2 + x^2 + y^2\, dx\, dy \\ &= \int_{x=-1}^{1}\int_{y=-1}^{1} -x^2 - y^2\, dx\, dy = -\frac{8}{3}. \end{aligned}$$

7. Redo for the computation for the cylinder above using the divergence theorem.

   The divergence is a constant 3. The volume of the sphere is 12π, and so we get 36π again, as expected.