18.02 Recitation
Problems
nn28 November 2011

1. *Consider the field $\vec{F} = \langle x^2, 2y, z\rangle$. Compute the flux across a square in space with vertices at $(\pm 1, \pm 1, 1)$ by integrating directly. How would the answer change if we changed the function for the first entry of $\vec{F}$?*

   For this one $d\vec{S}$ is just $\hat{k}\,dx\,dy$. So

$$\int_{x=-1}^{1}\int_{y=-1}^{1}\left\langle x^2, 2y, 1\right\rangle \cdot \langle 0,0,1\rangle\; dx\,dy = 4.$$

2. *Find the flux of $\vec{F} = \langle x, y, z\rangle$ across the unit sphere via direct computation, and without any computations.*

   Without computation, we see that $\vec{F}$ is parallel to $\hat{n}$, and both of these have magnitude 1, so $\vec{F}\cdot\hat{n}$ is a constant 1 and the flux is just the surface area of the sphere, which is 4π. With computation, use $dS = a^2 \sin\phi\, d\phi\, d\theta$, where $a = 1$. Then $\hat{n} = \langle x, y, z\rangle$, which is already a unit vector.

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \langle x,y,z\rangle \cdot \langle x,y,z\rangle \sin\phi\, d\phi\, d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi\, d\phi\, d\theta = \int_{\theta=0}^{2\pi} 2\, d\theta = 4\pi.$$

3. *Find the flux of $\vec{F} = \langle x^2 z, 2y, z\rangle$ across a cylinder whose base is a circle in the xy-plane of radius 3 and whole height is 2. Include both the side of the cylinder and its top and bottom faces.*

   We want to get both the sides of the cylinder and the top and bottom bits. We're going to need to integrate over all of them separately. For the sides of the cylinder, use $\hat{n} = \frac{1}{3}\langle x, y, 0\rangle$. Also $dS = 3\,dz\,d\theta$. So the integral is

$$\begin{aligned}
\text{flux} &= \int_{z=0}^{2}\int_{\theta=0}^{2\pi} \left\langle x^2 z, 2y, z\right\rangle \cdot \frac{1}{3}\langle x, y, 0\rangle\,(3d\theta\,dz)\\
&= \int_{z=0}^{2}\int_{\theta=0}^{2\pi} (x^3 z + 2y^2)\,d\theta\,dz\\
&= \int_{z=0}^{2}\int_{\theta=0}^{2\pi} ((3\cos\theta)^3 z + 2(3\sin\theta)^2)\,d\theta\,dz\\
&= \int_{z=0}^{2} 18\pi\,dz = 36\pi.
\end{aligned}$$

   We also need to integrate over the top and bottom of the cylinder. The top has $\hat{n} = \hat{k}$, and

$$\begin{aligned}
\text{flux} &= \int_{r=0}^{3} \int_{\theta=0}^{2\pi} \left\langle x^2 z, 2y, z \right\rangle \cdot \langle 0,0,1\rangle \; r \, d\theta \, dr \\
&= \int_{r=0}^{3} \int_{\theta=0}^{2\pi} z \, r \, d\theta \, dr = \int_{r=0}^{3} \int_{\theta=0}^{2\pi} 2 \, r \, d\theta \, dr \\
&= 18\pi
\end{aligned}$$

The bottom works the same way, except that z is 0 and so the flux is too. The total flux is $36\pi + 18\pi = 54\pi$.

4. *Redo for the computation for the cylinder above using the divergence theorem.*

   The divergence is $2xz + 2 + 1 = 2xz + 3$.

$$\begin{aligned}
\text{flux} &= \int_{z=0}^{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{3} (2(3\cos\theta)(z) + 3) \; r \, dr \, d\theta \, dz \\
&= \int_{z=0}^{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{3} (6rz\cos\theta + 3r) \; dr \, d\theta \, dz \\
&= \int_{z=0}^{2} \int_{\theta=0}^{2\pi} (27z\cos\theta + 27/2) \, d\theta \, dz \\
&= \int_{z=0}^{2} (0 + 54\pi z) \, dz = 54\pi.
\end{aligned}$$

5. *Find the flux of $\vec{F} = \langle x, y, z\rangle$ across the paraboloid $z = x^2 + y^2$ in the region over the square with vertices $(\pm 1, \pm 1)$. Interpret the sign of your answer.*

   Here $d\vec{S} = \langle -2x, -2y, 1\rangle \; dx \, dy$ from the formula. So

$$\begin{aligned}
flux &= \int_{x=-1}^{1} \int_{y=-1}^{1} \left\langle x, y, x^2 + y^2 \right\rangle \cdot \langle -2x, -2y, 1\rangle \; dx \, dy \\
&= \int_{x=-1}^{1} \int_{y=-1}^{1} -2x^2 - 2y^2 + x^2 + y^2 \, dx \, dy \\
&= \int_{x=-1}^{1} \int_{y=-1}^{1} -x^2 - y^2 \, dx \, dy = -\frac{8}{3}.
\end{aligned}$$

   Our normal vector points inward on the paraboloid, while the vector field is pointing outward from the origin, and therefore outward through the paraboloid. Thus, it's not unexpected that the flux is negative.

6. *Parametrize the following surfaces:*

   (a) *The front face of a $2 \times 2 \times 2$ cube centered at the origin.*

   (b) *The part of the unit sphere in the first octant.*

(c) *The face of a triangle with vertices at* $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

(d) *A the paraboloid* $z = 1 - x^2 - y^2$ *in the region where* z *is positive.*

(e) *A cylinder of radius* 2 *around the* z*-axis, bounded by* $z = 0$

and $z = 3$.

I'll say something about parametrized surfaces on Wednesday.