18.02 Recitation
Problems
05 December 2011

1. For what a, b, c will the field $\langle ay^2z, yz(bx+z), y^2(x+cz)-3z^2\rangle$ be conservative?

   The curl is computed as

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\hat{\imath} - (R_x - P_z)\hat{\jmath} + (Q_x - P_y)\hat{k} \\ &= ((2-b)xy + (2c-2)yz)\,\hat{\imath} + \big((1-a)y^2\big)\,\hat{\jmath} + ((b-2a)yz)\,\hat{k}\end{aligned}$$

   So $a = 1, b = 2, c = 1$.

2. For what a, b, c will the differential form $ay^2z\,dx + yz(bx+z)\,dy + (y^2(x+cz)-3z^2)\,dz$ be exact?

   This field being exact is equivalent to the vector field from the first question being conservative. So it will hold for the same values of a, b, c.

3. Is the field $\vec{F} = \langle 2xyz, x^2z+z, x^2y+y+1\rangle$ conservative?

$$\begin{aligned}\operatorname{curl}\vec{F} &= (R_y - Q_z)\hat{\imath} - (R_x - P_z)\hat{\jmath} + (Q_x - P_y)\hat{k} \\ &= ((x^2+1)-(x^2+1))\hat{\imath} + ((2xy-2xy))\hat{\jmath} + (2xz-2xz)\hat{k} = 0\end{aligned}$$

   So it is conservative.

4. If your answer to the previous question was "yes", find a function f with $\nabla f = \vec{F}$. (If your answer was "no", better check your work). Try to use both the line integral method and the partial derivative method.

   Let's do this both ways. For the first, suppose f is the potential function. $f_x = 2xyz$, so $f = x^2yz + g(y,z)$. This means $f_y = x^z + g_y$, which means $g_y = z$. so $g = yz + h(z)$, and $f = x^2yz + yz + h(z)$. At last we differentiate with respect to z: $f_z = x^2y + y + h_z$, so $h_z = 1$ and $h = z$. This at last gives our answer $x^2yz + yz + z$.

$$\begin{aligned}f(a,b,c) - f(0,0,0) &= \int_0^1 \big\langle 2xyz, x^2z+z, x^2y+y+1\big\rangle \cdot \langle a,b,c\rangle\ dt \\ &= \int_0^1 (2a^2bct^3 + a^2bct^3 + bct + a^2bct^3 + bct + c)\,dt \\ &= \frac{a^2bc}{2} + \frac{a^2bc}{4} + \frac{bc}{2} + \frac{a^2bc}{4} + \frac{bc}{2} + c \\ &= a^2bc + bc + c\end{aligned}$$

So $f(x,y,z) = x^2yz + yz + z$, which is the same answer we got before. Phew.

5. Let $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ be the twisted cubic from lecture. Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle x, y, z \rangle$, for $0 \le t \le 1$ directly.

   We have $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$ as our parametrization. Just plug these in.

$$\int_{t=0}^{1} (t + 2t^3 + 3t^5)\, dt = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

   Alternatively, observe that it's the gradient of $x^2/2 + y^2/2 + z^2/2$, and use the fundamental theorem.

6. Compute $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{r}$ is again the twisted cubic, and $\vec{F}$ is the field from number 3. Use the fundamental theorem of calculus for line integrals.

   It's just $f(1,1,1) - f(0,0,0) = 3 - 0 = 3$, since those points are the ends of the paths.

7. Parametrize the following paths in three dimensions:

   (a) A straight line from $(1,2,3)$ to $(4,5,7)$.
   $\vec{r}(t) = (1+3t, 2+3t, 3+4t)$, with $0 \le t \le 1$.

   (b) A helix of radius a, rotating with angular velocity 1, moving upwards at a rate of 2.
   $\vec{r}(t) = (a\cos t, a\sin t, 2t)$, t any real number.

   (c) A circle in the plane $x = -1$, of radius 4.
   $\vec{r}(t) = (-1, 4\cos t, 4\sin t)$ $(0 \le t \le 2\pi)$

   (d) A line of longitude on the sphere of radius 1 centered at the origin, passing through $(\sqrt{2}/2, \sqrt{2}/2, 0)$.
   It's pretty easy to see how to do this in spherical coordinates. We want $\theta = \pi/4$ for all t, $\rho = 1$ for all t, and $\phi = t$, to keep going around the line of longitude. Having the parametrization in spherical isn't useful for computing the line integral (how can we find $d\vec{r}/dt$), but we can convert this to spherical coordinates to get the required result: $\vec{r}(t) = (\rho\cos\theta\sin phi, \rho\sin\theta\cos\phi, \rho\cos\phi) = (\sqrt{2}/2\sin t, \sqrt{2}/2\cos t, \cos t)$.