18.02 Recitation
Problems
07 December 2011

1. (6F3) Verify Stokes when S is the rectangle from part a) and $\vec{F} = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$.

   This one's written up in the book.

2. Verify Stokes when S is the region from b) of the first problem (the part of a sphere of radius 2 lying below $z = 1$), C its boundary, and $F = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

   This field has curl 0 (easy to check using the formula). We can parametrize the path for the line integral via $\vec{r}(t) = (\sqrt{2}/2\cos t, \sqrt{2}/2\sin t, 1)$. Plugging this in to the equation for the line integral,

$$W = \int_0^{2\pi} \left\langle \sqrt{2}/2\sin t, \sqrt{2}/2\cos t, 1/2\sin t\cos t \right\rangle \cdot \left\langle -\sqrt{2}/2\sin t, \sqrt{2}/2\cos t, 0 \right\rangle\, dt$$
$$= \int_0^{2\pi} -\frac{1}{2}\sin^2 t + \frac{1}{2}\cos^2 t\, dt = 0.$$

   Remember what Stokes says:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F}) \cdot \hat{n}\, dS.$$

   We've just checked that both are 0.

3. Consider a cylinder of radius 1, bounded below by the plane $z = 0$ and above by $z = 2 + \cos 2\theta$. Let $\vec{F} = \langle -y, x, 0\rangle$. Use Stokes' theorem to compute the integral $\oint \vec{F} \cdot d\vec{r}$ over the top edge.

   Call the bottom edge C_2 and the top edge C_1. Stokes theorem says that

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS - \oint_{C_2} \vec{F} \cdot d\vec{r}.$$

   We'll compute both things on the right. The bottom edge is parametrized by $\langle \cos t, \sin t, 0\rangle$, where $\vec{F} = \langle -\sin t, \cos t, 0\rangle$. This gives $\vec{F} \cdot d\vec{r} = 1$, so $\oint_{C_2} \vec{F} \cdot d\vec{r} = 2\pi$. But $\nabla \times \vec{F} = \langle 0, 0, 2\rangle$, and so $\nabla \times \vec{F} \cdot \hat{n} = 0$, and $\iint_S \iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = 0$. So the integral we're after is $0 - 2\pi = -2\pi$.

4. Let S be a cylinder of height 2 and radius 1 centered at the origin (not including either of the ends!). The region has two boundary components. Find an orientation compatible with both of them, and verify Stokes' theorem for the region, with the field $\vec{F} = \langle yz, -xz, 0\rangle$.

   We want to go clockwise around the top edge and counterclockwise around the bottom edge. Along the top edge, $z = 1$ and the field is $\langle y, -x, 0\rangle$, which is parallel to $d\vec{r}$.

So $\int_{C_1} \vec{F} \cdot d\vec{r} = 2\pi$. On the bottom, the field is $\langle -y, x, 0\rangle$ since $z = -1$. Again it's parallel to $d\vec{r}$ since we're going the other direction. So this integral is 2π also. So $\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 4\pi$.

On the other hand the flux is $\langle x, y, 0\rangle$, so $\nabla \times \vec{F} \cdot \hat{n}\, dS$ is a constant 1. So $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = 4\pi$ is just the surface area of the cylinder.

5. Let S be a Möbius strip. How many boundary curves does it have? How should S be oriented? What about a strip twisted k times? Can you come up with an example of a closed surface (i.e. there are no boundary curves) which can't be oriented?

   It has two, but there's no sensible way to orient it (it only really has one side, so which way would you have point out?) There are non-orientable surface, but they don't fit inside $\mathbb{R}^3$ without intersecting themselves. Wiki "Klein bottle" if you haven't seen one of these thing.s

6. What does Stokes' theorem tell us in the case that $\nabla \times \vec{F} = 0$? Does this make sense?

   Stokes then predict that the integral $\oint_C \vec{F} \cdot d\vec{r} = 0$, for any closed curve C. This seems strange, but isn't as strange when you realize that the curl being 0 means the field is conservative, and so the integral around a closed loop is 0 by the fundamental theorem of calculus.