18.02 Recitation
Problems
12 December 2011

1. Find the equation for a plane passing through the point $(1,1,1)$ and containing the line given parametrically by $(1,0,0)+(1,0,1)t$. What angle does this plane make with the vector $(0,1,0)$?

   It goes through $(1,0,0)$ and contains vectors in the directions $(0,1,1)$ and $(1,0,1)$ from there. The cross product of these two will give the normal vector. You can compute it as $(1,1,-1)$. So the plane is $x+y-z=1$. The cosine of the angle of the vector in question with the normal vector is $\vec{1,1,-1} \cdot \langle 0,1,0\rangle / |\langle 1,1,-1\rangle|$ with the normal vector is $\cos^{-1}(1/\sqrt{3})$, so the angle with the plane is $\pi/2 - \cos^{-1}(1/\sqrt{3})$.

2. (2I-2) Using Lagrange multipliers, tell which point P in the first octant and on the surface $x^3y^2z = 6\sqrt{3}$ is closest to the origin.

   (This one's in the book)

3. Compute the work $\int_C \vec{F} \cdot d\vec{r}$ where C follows the parabola $y = x - x^2$ between $(0,0)$ and $(1,0)$, and $\vec{F} = \langle 2xy, x^2\rangle$. Can you do it via: direction calculation, Green's theorem, fundamental theorem for line integrals?

   Directly: $\vec{r}(t) = (t, t-t^2)$. So

   $$\int_{t=0}^{1} \left\langle 2t^2 - 2t^3, t^2 \right\rangle \cdot 1, 1-2t\, dt = \int_{t=0}^{1} 3t^2 - 4t^3\, dt = 0.$$

   Curl:

   $$\int_{x=0}^{1} \int_{y=0}^{x-x^2} 0\, dy\, dx = 0.$$

   and over the bottom $\vec{r}(t) = (t,0)$

   $$\int_{t=0}^{1} \left\langle 0, t^2 \right\rangle \cdot 1, 0\, dt = 0.$$

   So it's definitely 0.

4. Solve the linear system $x+y-z=0$, $-x+2y-z=0$, $2x+y+z=7$. Hint: the inverse matrix is of the form

   $$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -2 & 1 \\ -1 & 3 & 2 \\ -5 & 1 & x \end{pmatrix}$$

   Write it as

   $$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}.$$

Invert the matrix: $\frac{1}{7}\begin{pmatrix} 3 & -2 & 1 \\ -1 & 3 & 2 \\ -5 & 1 & 3 \end{pmatrix}$. So it's $(1,2,3)$.

So the solution is

$$\frac{1}{7}\begin{pmatrix} 3 & -2 & 1 \\ -1 & 3 & 2 \\ -5 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

It's $(1,2,3)$

5. Evaluate $\iint_S \vec{F}\cdot\vec{n}\,ds$, where S is the part of $z = 1 - x^2 - y^2$ lying above the plane $z = 0$, $\vec{F} = \langle zy, zx, 1+z\rangle$.

   Could do directly, but easier to use divergence theorem: $\operatorname{div} F = 1$, so $\int_{r=0}^{1}\int_{\theta=0}^{2\pi}(1 - r^2)r\,dr\,d\theta = \frac{\pi}{2}$. Over the other face it's equal to $\iint_{r=0}^{1}\iint_{\theta=0}^{2\pi} = \langle 0,0,1\rangle \cdot 0,0,1 = \pi$. So $? + (-\pi) = \pi/2$, and our answer is $3\pi/2$.

6. Consider the surface $x^2 + 4y^2 + 2z^2 = 1$. What is the tangent plane at $(1/2, 1/4, 1/2)$?

   The normal is $\langle f_x, f_y, f_z\rangle = \langle 2x, 8y, 4z\rangle = 1,2,2$, and so the plane is $x + 2y + 2z = 2$ (the constant on the right side of the equality being chosen to make sure that it goes through the point).

7. Let $f(x,y,z) = x + y + z$. Find $\left(\frac{\partial f}{\partial x}\right)\Big|_y$ subject to remaining on the above ellipsoid, at the point.

   Well, $df = dx + dy + dz$. We're going to have $dy = 0$, and want to express dx in terms of dz. That will be achieved by taking the differential of the constraint: $2x\,dx + 8y\,dy + 4z\,dz = 0$. If $dy = 0$, then at this point $dx + 2\,dz = 0$. So $df = dx/2$, and the derivative is $1/2$.

8. Consider the point $(1,1,1)$. What is this point in spherical coordinates? How fast does ρ change when x and z each increase at a rate of 1?

   Well $\rho = \sqrt{x^2 + y^2 + z^2}$. So

   $$\frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}.$$

   Thus

   $$\frac{\partial\rho}{\partial z} = \frac{x}{\sqrt{x^2+y^2+z^2}} + \frac{x}{\sqrt{x^2+y^2+z^2}}.$$

   At the point in question this is $2/\sqrt{3}$.

9. Find the area of the region bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $x/y = 0$, $x/y = 1$. Use a change of coordinates in a double integral.

   Take $u = x^2 + y^2$, $v = x/y$. Then

   $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} 2x & 2y \\ 1/y & -x/y^2 \end{pmatrix} = -2x^2/y^2 - 2 = -2v^2 - 2.$$

The absolute value is $2v^2 + 2$.

which is

$$0 \int_{v=0}^{1} \int_{u=1}^{4} \frac{1}{2v^2+2}\, du\, dv = 3\left(\frac{1}{2}\tan^{-1}(v)\Big|_{v=0}^{1}\right) = \frac{3\pi}{8}.$$