Concept	**Definition(s)**	**Applications**	**Examples**
Dot product	$\vec{a}\cdot\vec{b} = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\cos\theta$ $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$	Angle between vectors Component of vector in direction: $\text{comp}_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{\lvert\vec{v}\rvert}$	1B-6, 1B-11
Cross product	$\lvert\vec{a}\times\vec{b}\rvert = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin\theta$ Direction determined by right-hand rule $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$	Area of triangles and parallelograms Finding normal vectors to planes	1D-2, 1D-5
Determinant	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ 3×3 determinant: (sum of downward diagonals) − (sum of upward diagonals)	Inverting matrices Find area of parallelogram, volume of parallelpiped	1C-1, 1C-4
Scalar triple product	$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$	Volume of parallelpiped	1D-9, 1D-8
Matrix inverse	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ 3×3 inverse: see handout	Solving linear systems	1G-2, 1G-1
Matrix · vector	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$	Solving linear systems, transformations of the plane If $A\vec{x} = \vec{b}$, then $\vec{x} = A^{-1}\vec{b}$	1F-8b)
Matrix multiplication	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ Sizes must match! $= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$, etc.	Do one transformation, then another	1F-1, 1F-11
Normal vector to plane	$ax+by+cz=d$ normal to $\langle a,b,c\rangle$	Finding equation of plane, angle between planes, intersection of planes, …	1E-1, 1E-6
Parametrized line	$\ell(t) = \vec{x}_0 + \vec{v}t$	Describe line in space	1E-3, 1E-4
Equation of plane	$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ $(\vec{x}-\vec{x}_0)\cdot\vec{n}=0$	Describe plane in space Describe plane via point and normal vector	1E-2
Parametrizing a path	$x(t) = \cdot$, $y(t) = \cdot$	Describe path in plane/space (e.g. cycloid)	1I-6, 1I-7
Velocity, speed, etc.	$\vec{v}(t) = \frac{d}{dt}\vec{x}(t)$, $\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$ (differentiate each component), $\frac{ds}{dt} = \lvert\vec{v}(t)\rvert$, $T = \frac{\vec{v}(t)}{\lvert\vec{v}(t)\rvert}$	Physics (Kepler's laws)	1J-4, 1J-5
Geometry problems via vectors	all of the above!		1A-9, 1B-11, 1E-7

Contour lines/surfaces: (2A-1, 09/28 recitation problems)

- For a function $f(x, y)$ (resp. $f(x, y, z)$), these are defined by $f(x, y) = 0$ (resp. $f(x, y, z) = 0$)
- Perpendicular to gradient of f
- Be able to estimate the gradient and directional derivatives from a contour plot (e.g. topographic map)
- Tangent to contour line is a direction with zero directional derivative

Partial derivatives: (2A-2, 2B-5, 2B-9)

- $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$ = "What is the instantaneous rate of change of f when we change x?"
- Compute by treating y as a constant and pretending f is a function of the single variable x.
- Linear approximation:

$$f(x + \Delta x, y + \Delta y, z + \Delta z) \approx f(x, y, z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$$

Gradient: (2B-1, 2D-1)

- Defined by $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$
- It's a vector (which depends on x, y, and z)
- The tangent plane to a level surface $g(x, y, z)$ is perpendicular to the gradient: you can use this to find tangent planes

Directional derivative: (2D-2, 2D-4)

- For a direction vector $\hat{u}$, defined as the limit

$$\left.\frac{df}{ds}\right|_{\hat{u}} = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u})}{h}$$

- "What is the instantaneous rate of change of f when we change $\vec{x}$ in the direction of $\hat{u}$?"
- Compute it using $\left.\frac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}$.
- That's equal to $|\nabla f| \cos\theta$, where θ is the angle between ∇f and $\hat{u}$. So directional derivative is 0 if $\hat{u}$ is perpendicular to ∇f, maximum if $\hat{u}$ is in the direction of ∇f, and minimum if $\hat{u}$ is in the opposite direction.

Chain rule: (2E-1, 2E-5)

- If $f(u, v)$ is a function of u and v, with $u(x, y)$ and $v(x, y)$ are functions of x and y, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

  Same idea for other dependences – see the book.

Max-min problems: (2F-1, 2F-5)

- Maxima and minima occur at critical points, where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, or at the boundary.
- First step is to set up the problem: figure out the set of x and y over which you want to maximize.
- Find the critical points, check values there, and find max/min on the boundary (e.g. by parametrizing)

Least squares: (2G-1)

- This is an application of our max-min techniques.
- Given a bunch of data points, (x_i, y_i) (for $1 \le i \le n$), want to find a line (or other function) that approximates them as well as possible.
- The idea: think of the error $\sum_{i=1}^{n}(y - (ax_i + b))^2$ as a function of the parameters a and b, and then find the a and b for which this is minimized.

Second derivative test: (2H-1, 2H-5)

- Used to check if a critical point is a maximum or a minimum
- Compute
$$A = \frac{\partial^2 f}{\partial x^2}, \quad B = \frac{\partial^2 f}{\partial x\, \partial y}, \quad C = \frac{\partial^2 f}{\partial y^2}$$
- $AC - B^2 > 0, A > 0 \implies$ min, $AC - B^2 > 0, A < 0 \implies$ max, $AC - B^2 < 0 \implies$ saddle point, $AC - B^2 = 0 \implies$ can't say anything.

Lagrange multipliers: (2I-1, 2I-3)

- Want to maximize $f(x, y, z)$ subject to the condition $g(x, y, z) = c$.
- This will happen at a point where $\nabla f = \lambda \nabla g$ for some constant λ (i.e. where the two gradients are parallel to each other).

Non-independent variables: (2J-2, 2J-6)

- Derivative with constraints: subject to $g(x, y, z) = c$, how does f change when x changes?
- The constraint means that when we change x, y and/or z much change too in order to ensure that g stays constant.
- $\left(\frac{\partial f}{\partial x}\right)_z$ is the derivative we get when changing x, holding z constant, and changing y in order to keep $g(x, y, z) = c$.
- Find change in y in terms of change in x, while having 0 change in z: one way is to use $0 = dg = g_x\, dx + g_y\, dy + g_z\, dz$.
- Three ways to compute it: see Lecture Notes 14.

PDE

- A PDE is a differential equation involving the partial derivatives of a function, i.e. something like $f_{xx} + f_{yy} + f_{zz} = 0$.
- You should understand what such an equation means and be able to verify that a given function is a solution.

1. Double integrals (3A-2, 3A-3, 3D-5)

   (a) $\iint_R f(x,y)\,dA$ gives the volume under the graph of $f(x,y)$ in the region R. here dA is $dx\,dy$ or $r\,dr\,d\theta$, dependending on what coordinate system is appropriate.

   (b) Three steps to compute it: find the bounds of integration, evaluate the inner integral, evaluate the outer integral.

   (c) For a rectangle, bounds are:

   $$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

   (d) In general the bounds on the inner integral will depend on the outer variable. If x is between 1 and 2, and y is between x and x^2, the bounds are $\int_{x=1}^{2}\int_{y=x}^{x^2} f(x,y)\,dy\,dx$. (see notes 16 for some examples of setting this up)

   (e) The variable in the inner integral should have a range that depends on the outer variable (not the other way around!)

   (f) In evaluating the inner integral, treat the variable of the outer integral (x, in the above examples) as a constant, and integrate as usual. The result should be a function of x.

   (g) Then do the outer integral and you're done!

   Here are some particular double integrals worth keeping in mind:

Property	Integral
Area	$\iint_R 1\,dx\,dy$
Average value of function	(1/area) $\iint_R f(x,y)\,dx\,dy$
Center of mass: x coordinate	$1/M\ \iint_R x\delta(x,y)\,dx\,dy$
Center of mass: y coordinate	$1/M\ \iint_R y\delta(x,y)\,dx\,dy$
Moment of inertia about x-axis	$\iint_R \delta(x,y)y^2\,dx\,dy$
Moment of inertia about y-axis	$\iint_R \delta(x,y)x^2\,dx\,dy$
Polar moment of inertia	$\iint_R \delta(x,y)(x^2+y^2)\,dx\,dy$

2. Change of variables (3A-5, 3B-2, 3D-1, 3D-5)

   (a) Often an integral will be made easier by switching the order of integration. To do this, we need to reexpress the bounds on the same region but with the other order for the variables. Sketch the region, using the given bounds, and then figure out the bounds with the new order of integration.

   (b) You can also compute double integrals in polar coordinates. It works the same way: if the bounds on r depend on θ (the typical situation), then you should have dr as the inside integral, $d\theta$ as the outside one. An important thing to keep in mind is that the area element dA is expressed in polar as $r\,dr\,d\theta$. Don't forget the leading factor of r!

   (c) It's also useful to be able to change the coordinate system of an integral to new coordinates $u(x,y)$, $v(x,y)$. Usually a good first step is to figure out how to write x and y in terms of u and v. This will make the other steps easier.

   To do this, there are three basic steps:

      i. Express the bounds on the integral in terms of u and v. To do this, take the equation defining each edge of the region as an equality in terms of u and v.

      ii. Express the function $f(x,y)$ as a function of u and v.

      iii. Express the area element in terms of du and dv. This is done by writing

   $$du\,dv = |\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}|\,dx\,dy.$$

   The matrix $\left(\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right)$ is called the *Jacobian*. Don't forget to take the absolute value!

3. Vector fields (4A-1, 4A-3)

   (a) A vector field assigns to each point (x, y) a vector $\langle M(x,y), N(x,y)\rangle$.

   (b) Two important examples: $\vec{F}_1(x,y) = \langle x, y\rangle$. The vector at (x,y) points directly away from the origin. $\vec{F}_2(x,y) = \langle -y, x\rangle$ gives a field tangent to circles centered at the origin. Be sure to remember these!

   (c) One way to get a vector field is to start with a function $f(x,y)$, and take $\vec{F}(x,y)$ to be the gradient of the function at each point. A vector field that arises in this way is called a *gradient field*.

   (d) We can associate to a vector field two different functions, the divergence and the curl. These are defined by
   $$\operatorname{curl}\vec{F} = N_x - M_y, \qquad \operatorname{div}\vec{F} = M_x + N_y.$$

   (e) If a field is a gradient field, it's of the form $\langle \partial f/\partial x, \partial f/\partial y\rangle$. This implies that it has curl equal to 0.

   (f) The converse is almost true as well: if a field is defined on a simply connected region, and it has curl equal to 0, then it's a gradient field.

   (g) Be sure you know how to find the function of which it's a gradient (the potential function)! Two ways to do this are explained in Lecture Notes 21.

   (h) "Simply connected" just means the region has no holes in it. Examples are the whole plane, half plane, and unit disk. Non-examples are a plane with a hole in it, an annulus, or a plane with a line segment removed.

   (i) If the region isn't simply connected, a field can have curl zero but fail to be conservative. Be on the lookout whenever you see a field that looks like $\vec{F} = \big\langle -y/(x^2+y^2), x/(x^2+y^2)\big\rangle$.

4. Line integrals (4B-1, 4B-2, 4C-3, 4C-5)

   (a) Given a vector field $\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$, define the integral along a curve C parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$,
   $$\int_C \vec{F}\cdot d\vec{r} = \int_{t=a}^{b} \Big\langle \vec{F}(x(t), y(t))\Big\rangle \cdot \langle x'(t), y'(t)\rangle\; dt.$$

   (b) You'll also see this written out with $dx = x'(t)\,dt$, as
   $$\int_C \vec{F}\cdot d\vec{r} = \int_C M(x(t), y(t))\,dx + N(x(t), y(t))\,dy.$$

   (c) Either way, what you need to do is parametrize the curve, and plug in the $x(t)$ and $y(t)$ you use into the expression for the integral.

   (d) If the curve has "corners", you'll want to break the curve into smooth pieces, evaluate the integral on each of them separately, and then add the results together to get the total.

   (e) The fundamental theorem of calculus for line integrals states that if the vector field we're integrating is a gradient field $\vec{F} = \nabla f$, we can save a lot of trouble just by evaluating the function f at the endpoints of the path
   $$\int_C \vec{F}\cdot d\vec{r} = f(b) - f(a),$$
   where b is the endpoint of C, and a its starting point. In particular this means that the value of the integral doesn't depend on the path taken, so gradient fields are also called conservative fields.

5. Green's theorem (4D-1, 4D-5, 4D-7, 4E-3)

(a) Green's theorem lets us convert the line integral of a vector field around a closed loop to the integral of a related function over the inside of the region.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F}\, dA.$$

(b) Writing this out explicitly in terms of differentials,

$$\oint_C M\, dx + N\, dy = \iint_R N_x - M_y\, dy\, dx.$$

(c) If the curve C isn't a smooth curve, you'll need to break it into several pieces:

$$\oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F}\, dA.$$

If you can find the integral over two parts of the edge as well as the flux integral, you can use Green's theorem to find the third. This is helpful when there are two sides along which the integral is easy, and one where it's not.

(d) We can also consider the flux $\vec{F} \cdot \hat{n}$. This is a measure of how much the vector field "flows" across C.

(e) If your path is $\vec{r}(t)$, then $\hat{n}$ is obtained by rotating $d\vec{r}/dt$ 90 degrees to the left. Thus the normal is $\langle dy, -dx \rangle$, and we can use this to do the integral.

(f) To compute the flux across a curve, write out $\vec{F} \cdot \hat{n}$, plugging in $y'(t)\, dt$ for dy and $x'(t)\, dt$ for x in the expression $\langle dy, -dx \rangle$ for $\hat{n}$.

(g) There's a version of Green's theorem for flux, which says

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA.$$

(h) As differentials,

$$\oint_C M\, dy - N\, dx = \iint_R M_x + N_y\, dy\, dx.$$

This review sheet probably doesn't cover everything you need to know for the exam. Make sure you read through the lecture notes and know how to do all the problems on the practice exams! But this list is one place to start...

- Understand cylindrical and spherical coordinates and be able to convert points and functions between them. Certainly know (or better yet, be able to quickly derive) the formulas

Cylindrical	$x = r\cos\theta,\ y = r\sin\theta,\ z = z$
Spherical	$r = \rho\sin\phi,\ \theta = \theta,\ z = \rho\cos\phi$
Spherical	$x = \rho\cos\theta\sin\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\phi$

  You should be able to make the conversions in the other direction as well.

- (5A-1, 5A-2, 5B-1) Be able to set up bounds for triple integrals in all three coordinate systems! Don't forget the volume elements: $dx\,dy\,dz$, $r\,dr\,d\theta\,dz$, and $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$. The only way to get ready for this is to just do a few problems setting up bounds in the three coordinate systems we've discussed. At the very least, here are a few important ones and others to think about.

Rectangular	$\int_{x=a_1}^{a_2}\int_{y=b_1}^{b_2}\int_{z=c_1}^{c_2} dz\,dy\,dx$	Rectangular box
	$\int_{x=a_1}^{a_2}\int_{y=b_1}^{b_2}\int_{z=0}^{f(x,y)} dz\,dy\,dx$	Region under graph of $f(x,y)$, above the rectangle given in the bounds on x and y.
	...	Area under a plane, area between graphs of two functions, cylinder, sphere
Cylindrical	$\int_{z=c}^{d}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\,dr\,d\theta\,dz$	Cylinder of radius a, lying between $z=c$ and $z=d$
	$\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{f(r,\theta)} r\,dz\,dr\,d\theta$	Region under graph of $f(r,\theta)$
		Half cylinder, cone with vertex at origin, sphere, ...
Spherical	$\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{a} \rho^2\sin\phi\,dr\,d\theta\,d\phi$	Sphere of radius a
		Cone with vertex at origin, sphere passing through the origin, cylinder, some segment of a sphere, ...

- (5A-3, 5A-6, 5B-3) Triple integrals have a lot of the same uses that double integrals did. We can find average values of functions, volume, mass, center of mass, moment of inertia:

Property	Integral
Volume	$\iiint_R 1\,dV$
Average value of function	(1/vol) $\iiint_R f(x,y,z)\,dV$
Mass	$\iiint_R \delta(x,y,z)\,dV$
Center of mass: x coordinate	$1/M\ \iiint_R x\delta\,dV$
	(similar for other coordinates)
Moment of inertia about x-axis	$\iiint_R \delta(y^2+z^2)\,dV$
	(similar for other axes)

  When you get one of these, you need to choose the coordinate system in which to compute the integral, express the appropriate integrand in that coordinate system, and plug in the right volume element.

- (5C-3) One application new this unit was gravitation. The strength of the vertical component of the gravitational field from a mass on a particle at $(0,0,0)$ is given by

$$\vec{F} = \iiint_V \frac{G\delta}{\rho^3}\vec{\langle x,y,z\rangle}\cdot\vec{k}\,dV = G\iiint_V \frac{\cos\phi}{\rho^2}\delta\,dV.$$

Look in the lecture notes or supplemental notes G to read about gravitation (the bit in notes G is just a page plus some examples). Of particular note is Newton's theorem, which says that the gravititation attraction from a spherical planet of uniform density is the same as the attraction from a point mass at the center of the sphere. Look at lecture 26.

- (6A-1) We'll also look at vector fields in space. To specify a field, you need to give its $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ parts: $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, where each of these depends on x, y, z. Be able to visualize simple vector fields in space.

- (6B-4, 6B-6) We also define flux, analogously to the two-dimensional version. The integral

$$\text{flux} = \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \vec{F} \cdot d\vec{S}$$

measures the amount of flow of $\vec{F}$ across the surface S. Here $\vec{n}$ is a unit vector normal to the surface S (which will be different at different points), and dS is the area element. Note that $\hat{n}\, dS$ is sometimes written all together as $d\vec{S}$. To actually compute flux, you need to find $\hat{n}$ and dS for the appropriate surface. The most important surfaces to keep in mind are these:

Surface	$\hat{n}$	dS
Horizontal plane	$\hat{k}$	$dx\, dy$
Cylinder of radius a	$\frac{1}{a}\langle x, y, 0\rangle$	$a\, d\theta\, dz$
Sphere of radius a	$\frac{1}{a}\langle x, y, z\rangle$	$a^2 \sin\phi\, d\phi\, d\theta$
Graph of $f(x, y)$	$d\vec{S} = \langle -f_x, -f_y, 1\rangle$	

The integral for flux is a double integral, and the bounds of integration will involve two of the coordinates from your coordinate system. For example, if you want flux across a cylinder, you'll integrate for θ from 0 to 2π and z from a to b, while holding r fixed at the radius of the cylinder.

If your region is bounded by a couple different surfaces, you need to break the integral for flux into a few parts and add up the totals for each. To do the computation, take your field $\vec{F}$ (most likely expressed in terms of x, y, and z), and compute the dot product $\vec{F} \cdot \hat{n}$ (again depending on x, y, z). Then you need to write the resulting function in terms of the variables of integration (say, z and θ) and compute the double integral with the appropriate dS.

If the surface isn't on the list above, there's still hope. You need to parametrize it, so the surface is given by $(x(u, v),\, y(u, v))$. Set $\vec{r} = \vec{r}(u, v) = \langle x(u, v), y(u, v)\rangle$ and one checks that $d\vec{S} = \frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\, du\, dv$.

- (6C-5, 6C-6) There's a 3-dimensional version of the normal form of Green's theorem, the divergence theorem. It states that

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_R \operatorname{div} F\, dV$$

Here S is a closed surface and V is the region bounded by the surface. Note that $\vec{F}$ and its divergence should both be continuous here. The divergence of $\vec{F} = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k}$ is defined to be the function $\operatorname{div} F = P_x + Q_y + R_z$. The divergence theorem gives lets us compute the flux by doing a triple integral instead (or vice versa). Be sure to practice a couple problems where you compute the same integral both ways!

If you know something about electricity and magnetism, think of this as Gauss's law. Up to some constants, $\vec{F}$ is the electric field, and the divergence of $\vec{F}$ is the charge density.

Watch out for a shorthand here: setting $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$, it's common to write $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$.