(Solutions posted at `http://math.mit.edu/~johnl/1803/`)

1. Can you come up with a matrix A such that the system $\dot{\mathbf{u}} = A\mathbf{u}$ has a saddle, with an incoming ray solution along $(1, 1)$ and an outgoing one along $(1, 2)$?

2. Find the general solution to $y' + (\sin x)y = e^{\cos x}$.

3. (1.3.6) Consider the differential equation $\frac{dy}{dx} = x - y + 1$. Sketch some isoclines and describe the behavior of the integral curves. Which solutions have local maxima or minima? Points of inflection? Can you identify any fences? Estimate $y(10)$ for the solution with $y(0) = 2$.

4. Look at $\ddot{x} + 3\dot{x} + kx = k\cos(3t)$. What is the complex gain (regarding $\cos 3t$ as the input signal)? For what value of k is the amplitude of the system response maximized? How does the phase lag vary with k?

5. Let $f(t) = t \cdot |t|$. Express this function in terms of $u(t)$. What is $f'(t)$? $f''(t)$? $f'''(t)$? Sketch the last of these.

6. Consider the autonomous equation $\frac{dy}{dt} = (y - c)(y^2 - c)$, where c is a real number. What are the equilibrium solutions? Are they stable? (The answers to these questions depend on c). Sketch the bifurcation diagram.

7. Compute $\int e^t \cos t \, dt$.

8. Let A be the 2×2 matrix $\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$. Compute the determinant and trace, and describe type of the phase portrait for different values of a. What curve in the tr-det plane is traced out as a varies?

9. Find the unit impulse response for $D^2 + 7D + 12I$:
   (a) By solving $\ddot{x} + 7\dot{x} + 12x = 0$ with appropriate initial conditions.
   (b) Using Laplace transform.
   (c) By setting up the companion matrix and solving the 2×2 system using matrix exponential together with initial conditions.

10. Sketch the pole diagram for $F(s) = (s + 1)/(s^2 + 4s + 13)$. What does it tell you about the long-term behavior of the inverse transforms? Now invert the transform directly. Are these answers consistent?

11. Expand $3\cos(2t-\pi/3)$ as $a\cos(2t)+b\sin(2t)$. Write $\cos(3t)-\sin(3t)$ in standard form.

12. How would you solve $\ddot{x}+7\dot{x}+12x=q(t)$ where $q(t)=t^2$, $e^t\cos t$, te^t, $(t-1)e^t$, or e^{-3t}?

13. (a) A broken A/C unit has temperature $h(t)=\sin(\omega t)$, for some ω. Set up and solve a differential equation for the temperature of the room at time t. Is there any value of ω for which the temperature of the room is not bounded?

    (Just leave everything in terms of ω and the coupling constant k, along with a constant c. I trust you could find c given initial conditions).

    (b) After a repair the air conditioner has temperature given by a standard square wave. Give a formula for $f(t)$.

    (c) Suppose that the unit turns out at time $\pi/3$ instead of 0. Give a Fourier series for $f(t)$.

14. An operator has unit impulse response $w(t)=t^2u(t)$. What is $p(D)$? How can you solve $p(D)x=e^t$?

15. What is the Fourier series for $\sin^3(t)$?

16. Somehow I neglected to include a non-linear autonomous sytem in this list of problems. Please make sure you can do the ones on the practice tests!