

This is only a brief summary of the topics so far – be sure to study the notes as well!  
You can find old review sheets and practice tests on OCW.

## 1 Sinusoidal functions

A sinusoidal function is a function of the form  $f(t) = A \cos((2\pi/P)(t - t_0))$ . These are functions that look like cosine but have been scaled vertically and horizontally, and translated horizontally. Here  $A$  is the amplitude, the height of the maxima,  $P$  is the period, the time in which it repeats, and  $t_0$  is the time lag, the time of the first maximum. There are other names for other combinations of these numbers. The frequency is  $\nu = 1/P$ , the angular frequency is  $\omega = 2\pi/P$ , and the phase lag is  $\phi = \omega t_0$ . We'll usually write our sinusoids in the form  $A \cos(\omega t - \phi)$ .

Given the equation for a sinusoid, you should be able to draw a graph corresponding to the values of the parameters (or, given a graph, find the parameters). Be comfortable computing phase lag from time lag and  $\omega$ , etc.

Given a sinusoid  $A \cos(\omega t - \phi)$ , you can use the angle sum formula to write it as  $a \cos(\omega t) + b \sin(\omega t)$ . Here  $A = \sqrt{a^2 + b^2}$ ,  $a = A \cos \phi$ ,  $b = A \sin \phi$ . Using these equations you can go back and forth between the two forms. If you forget the equations, you can use the angle sum formula to figure them out.

One trick we can do with sinusoidal functions: to integrate  $e^{2t} \cos t$ , write it as  $\operatorname{Re} e^{(2+i)t}$ , integrate that, and then take the real part. Lecture notes 8 works this out in some detail.

## 2 Linear second order equations

The main equations we studied in this unit were of the form  $mx'' + bx' + kx = F_{ext}$ . This describes the movement of a mass on a spring, with a dashpot, subject to an external force. Exactly what  $F_{ext}$  will be and how it's related to the physical input is worth thinking about – see notes 13.

The solution will end up being something like  $c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{3} e^{-t} \cos(2t)$  with two terms with constants in front of them (this is the general solution to the homogeneous equation), and one term without (that's the particular solution). If you're given initial conditions, you can use them to solve for  $c_1$  and  $c_2$ . The basic procedure is this:

- Solve the homogeneous equation.
- Find a single solution to the inhomogeneous equation (i.e. a *particular solution*)

Add the two together, and you get a solution to the whole thing, by linearity. Sometimes the solutions to the homogeneous equation will all go to 0 as  $t$  goes to infinity. In this case every solution converges to a particular solution, called the steady state.

### 2.1 Solving the homogeneous equation

If we're looking at  $mx'' + bx' + kx = q(t)$ , first look at the associated homogeneous equation, namely  $mx'' + bx' + kx = 0$ . The thing to do here is look at the characteristic polynomial  $p(s) = ms^2 + bs + k$ . Find its zeroes (use the quadratic formula, or complete the square). Three things can happen, and I think it's worth remembering the general solution in each case.

- Two real roots  $r_1$  and  $r_2$ . In this case the general solution to the homogeneous equation is  $c_1 e^{r_1 t} + c_2 e^{r_2 t}$ . This case is called *overdamped*.
- Two complex roots  $a + bi$  and  $a - bi$  (you'll always get two conjugate complex numbers). In this case the general solution is  $c_1 e^{at} \cos bt + c_2 e^{at} \sin bt$ . This is the *underdamped* case.
- A single real root  $r$  (with multiplicity two). This is the *critically damped* case, and the general solution is  $c_1 e^{rt} + c_2 t e^{rt}$ .

To this we will add a particular solution to get a solution of our equation. Remember that except in the underdamped case, a solution can only cross 0 once. In the underdamped case, it will cross infinitely many times.

### 2.2 Finding a particular solution

How to find a particular solution depends on the form of the equation  $q(t)$ .

1.  $q(t) = Ae^{rt}$ . In this case, we'd guess that the solution should be  $ce^{rt}$  for some constant  $c$ , plug that in for  $x(t)$ , and solve for  $c$ . A shortcut is the *exponential response formula*, which tells you that  $c = A/p(r)$ , and a particular solution is given by  $Ae^{rt}/p(r)$ .

This isn't going to work if  $p(r) = 0$ . That's what's going to happen when  $ce^{rt}$  is already a solution of the homogeneous equation, so no choice of  $c$  can save you. This is an example of resonance. You need to use  $ct e^{rt}$  in this case, and some twist on the exponential response formula says that in fact  $c = A/p'(r)$ , and the particular solution is  $Ate^{rt}/p'(r)$ . If  $p'(r) = 0$  too, you're have to use  $t^2 e^{rt} \dots$ . For what to do then, I leave you to consult the lecture notes.

2.  $q(t) = c \cos(\omega t)$  (or  $\sin(\omega t)$ ). Here we use complex replacement and the exponential response formula (the combination of which is the sinusoidal response method).

First make a complex replacement: look at the complex equation  $mz'' + bz' + kz = Ae^{i\omega t}$ . The real part of a solution to this will solve the original equation. This we know how to solve using the exponential response formula. It's solved by  $Ae^{i\omega t}/p(i\omega)$ . Now we want to find the real part of this, which gives our solution. There are two options: multiply by the conjugate of  $p(i\omega)$ , expand  $e^{i\omega t}$  using Euler's formula, and take the real part; or, write  $A/p(i\omega)$  in polar coordinates and then multiply by  $e^{i\omega t}$ . You'll get the same answer either way.

The *complex gain* is  $z_p/y_{cx}$ , the factor by which the input is multiplied to get the output. This is denoted  $H(\omega)$ . Let  $h$  be the magnitude of  $H(\omega)$  (this is the *gain*) and  $\phi$  the negative of the argument of  $H(\omega)$  (this will turn out to be the phase lag of the solution), so that  $H(\omega) = he^{-i\phi}$ .

3.  $q(t) = at^2 + bt + c$ , or some other polynomial. Here you should guess that the particular solution is also going to be a polynomial. Plug in  $At^2 + Bt + C$  to the equation, and then solve for the values of  $A, B, C$  that will actually make it a solution. This is the *method of undetermined coefficients*.
4.  $q(t) = p(t)e^{rt}$ , where  $p(t)$  is a polynomial. Here the method to use is *variation of parameters*. If you were to plug in  $x(t) = u(t)e^{rt}$ , then  $mx'' + bx' + kx$  is going to be something times  $e^{rt}$ . So, for a judicious choice of  $u$ , we ought to get a particular solution. To find the  $u$  that actually works, plug in  $x(t) = u(t)e^{rt}$  to  $mx'' + bx' + kx$  and expand  $x''$  and  $x'$  using the product rule. The result will be a differential equation for  $u$ , which is probably easier to solve. Solve it, and then your particular solution is  $x_p(t) = u(t)e^{rt}$ .

This method is applicable to cases where you have something times  $e^{rt}$  on the right side. You did an example on the homework where it was  $e^{rt} \cos t$ , which can also be done by other means.

5.  $q(t)$  is of one of the above types, but has a time shift (e.g.  $q(t) = \cos(t - \pi/8)$ ). Here the thing to do is use the time-shifting principle: find a solution with  $q(t) = \cos(t)$ , and then substitute in  $t - \pi/8$  everywhere you see a  $t$ . This is a very useful thing for sinusoids that aren't just cosine: if you want to solve  $mx'' + bx' + kx = \cos(\omega t - \phi)$ , you can just solve with  $\cos(\omega t)$  on the right, then time shift (write  $\cos(\omega t - \phi) = \cos(\omega(t - t_0))$ ). Every sinusoid can be written in the previous form, so this works very generally.
6.  $q(t)$  is a sum of functions of several of the types above (e.g.  $q(t) = \cos(2t) - mg$ ). If you can find a solution for each one of the pieces that appears on the right, then the sum of all these things is a particular solution of the entire equation (this is the *principle of superposition*).

### 3 Other stuff

Keep in mind the notation of operators.  $D$  denotes differentiation with respect to  $t$ . If  $p(s)$  is a polynomial, say  $s^2 + 3s + 2$ , then  $p(D)$  is an operator, with  $p(D)x = x'' + 3x' + 2x$ .

Much of what we've said here works for higher-order equations as well:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_1 x' + a_0 x = q(t).$$

If the coefficients are constants, we can use the same approach: write down the characteristic polynomial. The exponential response formula and other methods above will carry over to the higher-order case as well.

Another trick is *reduction of order*. If you have an equation that only involves derivatives (as opposed to  $x$  itself), you can set  $u = x'$  and you'll get a differential equation for  $u$  of smaller order, which ought to be easier to solve.

Be ready for questions about how the gain and phase lag will depend on  $\omega$  if there is a sinusoidal input. There are examples in notes 16 and recitation notes 11.