This is only a brief summary of the topics so far – be sure to study the notes as well!
You can find old review sheets and practice tests on OCW.

1 Fourier series

The functions $\cos(nt)$ and $\sin(nt)$ have period $2\pi/n$. Fourier series let us write any reasonable function with period 2π as a (possibly infinite) combination of these basic functions:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

Here "reasonable" means f and its derivative are both piecewise continuous and averaged (the latter of these meaning that at each discontinuity, the function has $f(a) = (f(a+) - f(a-))/2$). The coefficients can be computed using the integrals

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

This all works for functions of period other than 2π too. If the period is $P = 2L$, the series will have terms $a_n\cos(n(\pi/L)t)$ and $b_n\sin(n(\pi/L)t)$, and there are similar integrals to compute the coefficients.

Actually computing any of these is integrals is a pain, especially on an hour exam. So make as many simplifying observations as you can. If f is odd, then its Fourier expansion will have only sin's in it ($a_n = 0$). If f is even, the expansion has only cos's ($b_n = 0$). The other important strategy is to try to relate our function to one for which the series is already known, by using translations, horizontal rescalings, differentiation, etc. If you know the Fourier series for the original function, you can just perform each of the operations on the Fourier series. The trickiest is dealing with translations: to get the series for $f(t-\pi/6)$ from that for $f(t)$, just substitute to get terms like $3\cos(4(t-\pi/3))$. A term like that isn't allowed in a Fourier series, but you can expand using the sum identity for cosine: $3\cos(4t)\cos(4\pi/3) - 3\sin(4t)\sin(4\pi/3)$, and this is allowed. Expand all the terms that way and you get the Fourier series.

Just about the only function we actually computed the series for was the square wave, which is odd, of period 2π, and takes the value 1 between 0 and π (this data is enough to determine it completely).

$$\mathrm{Sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right)$$

Fourier series turn out to be very useful for differential equations. Here's an "example". If you have a spring system with any periodic external force $q(t)$, it's described by the equation $mx'' + bx' + kx = q(t)$. Expand $q(t)$ as a Fourier series:

$$mx'' + bx' + kx = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots.$$

Each term can be handled one by one: $mx'' + bx' + kx = b_2\sin(2t)$ is something you know how to solve using the sinusoidal response method. Find the solution for each term in the Fourier series for $q(t)$ and add them all up. By superposition, this gives a solution to our original equation.

2 Delta functions etc.

We think of the delta function as a function that has very large values when t is very close to 0. It's so large, in fact, that $\int_{-\infty}^{\infty}\delta(t)\,dt = 1$. The Heaviside step function $u(t)$ is the function which is 0 for $t < 0$ and 1 for $t > 0$. Its derivative is the delta function (really all of this should be interpreted in terms of time scales, for which see notes 23). A *regular function* is piecewise smooth, i.e. it's broken into a bunch of pieces and infinitely differentiable on each of them. A *singularity* function is a sum of delta functions. A *generalized function* is the sum of a regular function and a singularity function. To take the derivative of a piecewise function, here's what to do: take the usual derivative of the function of the function everywhere except where the function jumps. If the function jumps by some amount $J = f(a+) - f(a-)$ at $t = a$, then throw in a $J\cdot\delta(t-a)$. To multiply a delta function by a continuous function, don't forget the rule $f(t)\cdot\delta(t-a) = f(a)\cdot\delta(t-a)$. With all these definitions, the product rule and fundamental theorem of calculus both work.

The delta function and unit step function are both useful in setting up differential equations to describe real-life situations. Here are the two basic examples. $x' - kx = q(t)$ describes a bank account, paying interest at rate k, and such that money is deposited at a rate of $q(t)$. If $q(t) = a\cdot u(t)$ is the step function, this means you deposit no money until $t = 0$, and then deposit at a constant rate of 1 for $t > 0$. If $q(t) = a\cdot\delta(t)$, this means we deposit no money for $t < 0$, at $t = 0$ we make a lump-sum deposit of a, and then never deposit money again. In particular the balance is discontinuous and jumps by a at $t = 0$. The second order example is a driven mass-spring system $mx'' + bx' + kx = F_{ext}(t)$. If $F_{ext}(t) = u(t)$, there's no force until $t = 0$, at which time a force of 1 begins. If $F_{ext}(t) = \delta(t)$, this means the mass sits until $t = 0$, at which time it is struck very hard. The derivative $x'(0)$ jumps by $1/m$ at $t = 0$.

	First order		Second order	
	Step	Impulse	Step	Impulse
Equation:	$cx' + kx = u(t)$	$cx' + kx = \delta(t)$	$mx'' + bx' + kx = u(t)$	$mx'' + bx' + kx = \delta(t)$
For $t > 0$ solve:	$cx' + kx = 1$	$cx' + kx = 0$	$mx'' + bx' + kx = 1$	$mx'' + bx' + kx = 0$
With conds:	$x(0) = 0$	$x(0) = 1/c$	$x(0) = 0,\ x'(0) = 0$	$x(0) = 0,\ x'(0) = 1/m$
Don't forget to multiply by $u(t)$ at the end!				

The unit step response for an operator $p(D)$ is the function $x(t)$ that satsifies $p(D)x = u(t)$, starting with initial conditions $x^{(n)}(0-) = 0$. The unit step response is a solution to $p(D)x = 1$, with all initial conditions 0 (for first order, this means $x(t) = 0$, for second order, $x(t) = x'(t) = 0$). You know how to solve $p(D)x = 1$ from last unit: solve the homogeneous equation, find a particular solution, add them). Don't forget to multiply your solution by $u(t)$, since you want it to be 0 for $t < 0$.

The unit impulse response for an operator $p(D)$ solves $p(D)x = \delta(t)$. For $t > 0$ this is a solution to the homogeneous equation $p(D)x = 0$ with initial conditions all 0, except the highest-order derivative has $x^{(n)}(0) = 1/a_n$, where a_n is the leading coefficient of the operator (for the bank example: $x(t) = 1$; for the spring example: $x(t) = 0$, $x'(t) = 1/m$). Often the unit impulse response is denoted $w(t)$. You can also find the step and impulse response using Laplace transform. More on this later.

Given two functions $f(t)$ and $g(t)$, the convolution is defined by the integral

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

This is another function of t. Key fact: convolution commutative, so $f(t) * g(t) = g(t) * f(t)$. The reason we care about this is that if $w(t)$ is the weight function above, then a solution to $p(D)x = q(t)$ with rest initial conditions is given by $q(t) * w(t)$. This means that if you know the weight function for an operator, you can get a solution to $p(D)x = q(t)$ by computing the convolution integral (however in most cases doing this integral will be harder than just solving the equation directly). Keep in mind that if are you trying to convolve two functions, it's often good to take "g" to be the simpler one, so you'll have fewer τ terms involved in the integral.

3 Laplace transform

Given a function $f(t)$ (satisfying some conditions that I won't worry about), the Laplace transform is a function $F(s)$, defined by $\mathcal{L}[f] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$. It satisfies a bunch of useful relations. No need to memorize these: you'll get a table on the exam.

Linearity	$\mathcal{L}[a\,f(t) + b\,g(t)] = a\,F(s) + b\,G(s)$	$\mathcal{L}[1] = \frac{1}{s}$
s-shift	$\mathcal{L}[e^{rt}f(t)] = F(s-r)$	$\mathcal{L}[\delta(t-a)] = e^{-a}$
t-shift	$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{at}] = 1/(s-a)$
s-derivative	$\mathcal{L}[tf(t)] = -F'(s)$	$\mathcal{L}[t^n] = n!/s^{n+1}$
t-derivative	$\mathcal{L}[f'(t)] = s\,F(s) - f(0-)$	$\mathcal{L}[\cos(\omega t)] = s/(s^2+\omega^2)$
	$\mathcal{L}[f''(t)] = s^2F(s) - s\,f(0-) - f'(0-)$	$\mathcal{L}[\sin(\omega t)] = \omega/(s^2+\omega^2)$
Convolution	$\mathcal{L}[f(t) * g(t)] = F(s)\,G(s)$	$\mathcal{L}[t\,\cos(\omega t)] = 2\omega s/(s^2+\omega^2)^2$
Weight fct	$\mathcal{L}[w(t)] = W(s)$	$\mathcal{L}[t\,\sin(\omega t)] = (s^2-\omega^2)/(s^2+\omega^2)^2$

The important method here is that given a differential equation, you can hit both sides with the Laplace transform and it turns into an algebraic problem. You can often solve for the Laplace transform $F(s)$ of the solution, and then try to find a function having the given trantsform (this is usually the hard part). Note that taking the transform of f' and f'' is where the initial conditions come into play: the transform of derivatives of f depends on $f(0-)$, $f'(0-)$, ... by the t-derivative rule. Then you solve for $F(s)$ and try to invert the transform.

To find the impulse response $p(D)w = \delta(t)$, taking the transform yields $p(s)W(s) = 1$, so $p(s) = 1/W(s)$. This $W(s)$ is called the transfer function. If you know the transfer function, you can find the weight function by taking inverse transform. Laplace transform also lets you find the operator given the weight function: compute the transform of $w(t)$, find its reciprocal $1/W(s) = p(s)$, and the result is the characteristic function of the operator!

$W(s)$ is called the transfer function. Often it's useful to draw the "pole diagram" of $W(s)$, or more generally any Laplace transform $F(s)$. This just means to draw the complex plane and put a dot at every point where $W(s)$ is infinite (usually $W(s)$ is a rational function, so the dots go at the zeroes of the polynomials in the denominator). The pole diagram of a function $F(s)$ tells you quite a bit about the behavior of $f(t)$. Very roughly, a pole at $a + bi$ tells you your function has a term that acts like $e^{at}\cos(bt)$. For large t, all of these summands will be negligible except the one for which the real part a is largest, since the other exponentials decay more quickly. So the long term behavior is governed by by pole that is the rightmost in the diagram. If all poles have negative real part, $f(t)$ decays to 0, ...