18.03 Final Exam Review

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Unit 1

The main topic so far has been first order ODEs: $y' = f(x, y)$. We've developed techniques to qualitatively study their behavior by drawing a slope field and to numerically approximate a solution using Euler's method. We've also studied some particular classes of equations where additional methods make it possible to find exact solutions.

General methods

(1.3.3, 1.3.6; sketch and identify fences, 6.14)

Given any old first order ODE, it's usually not possible give an equation for the solution(s). But it's still possible to say quite a bit about them, either by sketching an approximation of the graph, or approximating the values of a solution at many points.

The basic technique to deal with these things is drawing a direction field (also known as a slope field), which gives a visual representation of solutions. Things to know:

1. Draw a *direction field* by sketching a few *isoclines*. Sketch integral curves after you've drawn the field.

2. Be able to prove that a particular isocline is a *fence* (there are examples of this in the first homework and the second recitation notes). Identify a *funnel* and explain its implications for the asymptotics of solutions.

3. A *separatrix* is a solution such that nearby solutions have dramatically different behavior.

There is one other tool we have that works for any first-order equation (rather than just those having some specific form): Euler's method. This gives a way to numerically approximate the solution to an equation. The basic mechanism is to start with a point (x_0, y_0). Increase x by the step size h should cause an increase in y of about $y'(x_0) \cdot h = f(x_0, y_0) \cdot h$. In general we move from (x_k, y_k) to (x_{k+1}, y_{k+1}) using the rule

$$x_{k+1} = x_k + h, \quad y_{k+1} = y_k + f(x_k, y_k) \cdot h.$$

Things will be easier to keep track of if you put all these numbers in a table. It's probably worth doing an example for practice.

Some classes of equations

(separable: 1.4.16, 1.4.34; autonomous: 1.7.5, 1.7.10)

The easiest case is when you have a *separable equation*: this is when $y' = f(x)g(y)$, and so you can write $dy/g(y) = f(x)\,dx$, integrate both sides, and then solve for y to get the general solution.

Another important class of equations is the *autonomous equations*, in which $f(x, y) = g(y)$ doesn't depend on x at all: this means that the change in y only depends on its current value. These are separable, but often the integrals involved in solving them are messy at best. On the other hand, it's easy to give simple qualitative descriptions of the solutions.

The *equilibria* are the values of y for which $g(y) = 0$, i.e. $f'(x, y) = 0$. A solution will generally increase or decrease until it reaches one of the equilibria, or else increase or decrease without limit.

The way to see what happens is to draw the *phase line*. This shows in which ranges of y the solution will increase or decrease, and thus which equilibrium it will tend towards. When drawing the phase line, it can help to draw a plot of $g(y)$ against y; then you can see for which ranges of y the derivative is either positive or negative. It also lets you identity the stable, unstable, and semistable equilibria: know how to identify and interpret these!

We have two special classes of autonomous equations which model some kind of population growth.

- Natural growth: $y' = k_0 y$.
- Logistic growth: $y' = k_0(1-(y/p))y$. The growth will slow as y approaches the *carrying capacity* p, since this means y' goes to 0. It is possible to give an exact solution, but often the qualitative behavior is all we care about.

If an equation involves a parameter, say a, the equilibria will depend on the value of a (you solve $y' = 0$ for y, and the answers will depend on a). A *bifurcation diagram* illustrates this dependence; draw a plane with a on the horizontal axis, y on the vertical, and graph the equilibrium values as functions of a (there will be more than one for each a value).

First order linear equations

(1.7.5, 1.7.10)

These are equations that only depend on x and x', and do so in a particularly simple way: they can be written in standard form $r(t)\,x'(t) + p(t)\,x(t) = q(t)$. Divide through by r to get *reduced standand linear form* $x'(t) + p(t)\,x(t) = q(t)$. This is a class of equations for which we can often find exact solutions.

Three important examples are computing compound interest with continuous depositing, simple RC circuits, and Newtonian cooling. You can read how to set these up in lecture notes 4 and 5.

There's a standard procedure for solving a first-order linear equation:

1. Solve the *associated homogeneous equation* $x'(t) + p(t)x(t) = 0$ (ignore q, basically). This is a separable equation: solve it, and you get $x_h = Ce^{-\int p(t)\,dt}$.

2. Once you solve the homogeneous equation you can modify it to find the solution to the inhomogeneous form. There are two approaches here, which are more or less equivalent.

   (a) Method one (variation of parameters): assume the general solution is ux_h, where x_h is some solution to the homogeneous equation. Plugging everything in gives $u = \int x_h(t)^{-1} q(t)\, dt$. Compute that integral, and then ux_h is your solution (details in notes #5).

   (b) Method two (integrating factor; you're going to get the same answer either way). Multiply through be $1/x_h$. Recognize the thing on the left as a derivative $\frac{d}{dt}(\text{something}(t) \cdot y)$, integrate both sides, and then solve for y.

Sometimes the language of systems and signals clarifies exactly what's going on in these situations. Think of the function $q(t)$ as an input, upon which the solution $y(t)$ somehow depends (it's the output). Initial conditions should be considered too. There's a sort of diagram you can draw – see lecture notes 4.

Complex numbers

Complex numbers have the form $a + bi$, where $i = \sqrt{-1}$. a is called the *real part*, b the *imaginary part*. You can multiply $(a + bi) \cdot (c + di)$ by using "foil" and remembering that $i^2 = -1$. The conjugate of $z = a + bi$ is the complex number $\bar{z} = a - bi$.

It's often useful to express complex numbers in polar coordinates. This works just like converting between polar and rectangular coordinates did in 18.02. r is called the modulus (or magnitute), and θ's called the argument). The key fact here is Euler's formula, which defines how to raise e to a complex power: $e^{i\theta} = \cos\theta + i \sin\theta$, so $e^{a+bi} = e^a(\cos b + i \sin b)$. Note that this need not be purely real or purely imaginary, even if θ is. Multiplying in polar coordinates is simple, as the moduli multiply and the arguments add: $(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$. This observation also lets you find the n^{th} roots of complex numbers. You can use this to give slick proofs of pretty much every trig identity. There's an example on the second problem set.

Unit 2

Sinusoidal functions

A sinusoidal function is a function of the form $f(t) = A\cos((2\pi/P)(t - t_0))$. These are functions that look like cosine but have been scaled vertically and horizontally, and translated horizontally. Here A is the amplitude, the height of the maxima, P is the period, the time in which it repeats, and t_0 is the time lag, the time of the first maximum. Other constants which derive from these are the frequency $\nu = 1/P$, angular frequency $\omega = 2\pi/P$, and phase lag $\phi = \omega t_0$. We'll usually write our sinusoids in the form $A\cos(\omega t - \phi)$, the "standard form".

Given the equation for a sinusoid, you should be able to draw a graph corresponding to the values of the parameters (or, given a graph, find the parameters). Be comfortable computing phase lag from time lag and ω, etc.

Given a sinusoid $A\cos(\omega t - \phi)$, you can use the angle sum formula to write it as $a\cos(\omega t) + b\sin(\omega t)$. Here $A = \sqrt{a^2 + b^2}$, $a = A\cos\phi$, $b = A\sin\phi$. Using these equations you can go back and forth between the two forms. If you forget the equations, you can use the angle sum formula to derive them.

One trick we can do with sinusoidal functions: to integrate $e^{2t}\cos t$, write it as $\operatorname{Re} e^{(2+i)t}$, integrate that, and then take the real part. Lecture notes 8 works this out in some detail.

Linear second order equations

The main equations we studied in this unit were of the form $mx'' + bx' + kx = F_{ext}$. This describes the movement of a mass on a spring, with a dashpot, subject to an external force. Exactly what F_{ext} will be and how it's related to the physical input is worth thinking about – see notes 13.

The solution will end up being something like $c_1e^{-2t} + c_2e^{-3t} + \frac{1}{3}e^{-t}\cos(2t)$ with two terms with constants in front of them (this is the general solution to the homogeneous equation), and one term without (that's the particular solution). If you're given initial conditions, you can use them to solve for c_1 and c_2. The basic procedure is this:

- Solve the homogeneous equation.
- Find a single solution to the inhomogeneous equation (i.e. a *particular solution*)

Add the two together, and you get a solution to the whole thing, by linearity. Sometimes the solutions to the homogeneous equation will all go to 0 as t goes to infinity. In this case every solution converges to a particular solution, called the steady state.

Solving the homogeneous equation

If we're looking at $mx'' + bx' + kx = q(t)$, first solve the associated homogeneous equation, namely $mx'' + bx' + kx = 0$. The thing to do here is look at the characteristic polynomial $p(s) = ms^2 + bs + k$. Find its zeroes (use the quadratic formula, or complete the square). Three things can happen, and I think it's worth remembering the general solution in each case.

- Two real roots r_1 and r_2. In this case the general solution to the homogeneous equation is $c_1e^{r_1t} + c_2e^{r_2t}$. This case is called *overdamped*.
- Two complex roots $a+bi$ and $a-bi$ (you'll always get two conjugate complex numbers). In this case the general solution is $c_1e^{at}\cos bt + c_2e^{at}\sin bt$. This is the *underdamped* case.
- A single real root r (with multiplicity two). This is the *critically damped* case, and the general solution is $c_1e^{rt} + c_2te^{rt}$.

To this we will add a particular solution to get a solution of our equation. Remember that except in the underdamped case, a solution can only cross 0 once, or not at all. In the underdamped case, it will cross infinitely many times.

Finding a particular solution

How to find a particular solution depends on the form of the equation $q(t)$.

1. $q(t) = Ae^{rt}$. In this case, we'd guess that the solution should be ce^{rt} for some constant c, plug that in for $x(t)$, and solve for c. A shortcut is the *exponential response formula*, which tells you that $c = A/p(t)$, and a particular solution is given by $Ae^{rt}/p(r)$.

   This isn't going to work if $p(r) = 0$. That's what happens when ce^{rt} is already a solution of the homogeneous equation, so no choice of c can save you. This is an example of resonance. You need to use cte^{rt} as your guess is this case, and some twist on the exponential response formula says that the right value is $c = A/p'(r)$, and the particular solution is $Ate^{rt}/p'(r)$. The gain will be maximized at $\omega_r = \sqrt{k/m}$, the natural frequency. If $p'(r) = 0$ too, you'll have to use $t^2e^{rt}\ldots$. For what to do then, I leave you to consult the lecture notes.

2. $q(t) = c\cos(\omega t)$ (or $\sin(\omega t)$). Here we use complex replacement and the exponential response formula (the combination of which is the sinusoidal response method).

   First make a complex replacement: look at the complex equation $mz''+bz'+kz = Ae^{i\omega t}$. The real part of a solution to this will solve the original equation. This we know how to solve using the exponential response formula. It's solved by $Ae^{i\omega t}/p(i\omega)$. Now we want to find the real part of this, which gives our solution. There are two options: multiply by the conjugate of $p(i\omega)$, expand $e^{i\omega t}$ using Euler's formula, and take the real part; or, write $A/p(i\omega)$ in polar coordinates and then multiply by $e^{i\omega t}$. You'll get the same answer either way.

   The *complex gain* is z_p/y_{cx}, the factor by which the input is multiplied to get the output. Here z_p is the complex particular solution you get, and y_{cx} is the complexification of the input signal. Note that this definition of gain is dependent on what we regard as the "input signal" – y_{cx} isn't necessarily just whatever appears on the right side of the equation. See [5] of lecture notes 16 for an example of this; the "b" on the right isn't regarded as part of y_{cx}. The complex gain is denoted $H(\omega)$ The magnitude of $H(\omega)$ is the gain h , and the negative of the argument of $H(\omega)$ is the phase lag ϕ, so that $H(\omega) = he^{-i\phi}$.

3. $q(t) = at^2+bt+c$, or some other polynomial. Here you should guess that the particular solution is also going to be a polynomial. Plug in $At^2 + Bt + C$ to the equation, and then solve for the values of A, B, C that will actually make it a solution. This is the *method of undetermined coefficients*.

4. $q(t) = p(t)e^{rt}$, where $p(t)$ is a polynomial. Here the method to use is *variation of parameters*. If you were to plug in $x(t) = u(t)e^{rt}$, then $mx'' + bx' + kx$ is going to be something times e^{rt}. So, for a judicious choice of u, we ought to get a particular solution. To find the u that actually works, plug in $x(t) = u(t)e^{rt}$ to $mx'' + bx' + kx$ and expand x'' and x' using the product rule. The result will be a differential equation for u, which is hopefully easier to solve. Solve it, and then your particular solution is $x_p(t) = u(t)e^{rt}$.

This method is applicable to cases where you have something times e^{rt} on the right side. You did an example on the homework where it was $e^{rt}\cos t$, which can also be done by complex replacement + ERF.

5. $q(t)$ is of one of the above types, but has a time shift (e.g. $q(t) = \cos(t - \pi/8)$). Here the thing to do is use the time-shifting principle: find a solution with $q(t) = \cos(t)$, and then substitute in $t - \pi/8$ everywhere you see a t. This is a very useful thing for sinusoids that aren't just cosine: if you want to solve $mx'' + bx' + kx = \cos(\omega t - \phi)$, you can just solve with $\cos(\omega t)$ on the right, then time shift (write $\cos(\omega t - \phi) = \cos(\omega(t - t_0))$). Every sinusoid can be written in the previous form, so this works very generally.

6. $q(t)$ is a sum of functions of several of the types above (e.g. $q(t) = \cos(2t) - mg$). If you can find a solution for each one of the pieces that appears on the right, then the sum of all these things is a particular solution of the entire equation (this is the *principle of superposition*).

Other stuff

Keep in mind the notation of operators. D denotes differentiation with respect to t. If $p(s)$ is a polynomial, say $s^2 + 3s + 2$, then $p(D)$ is an operator, with $p(D)x = x'' + 3x' + 2x$.

Much of what we've said here works for higher-order equatinos as well:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_1 x' + a_0 x = q(t).$$

If the coefficients are constants, we can use the same approach: write down the characteristic polynomial. The exponential response formula and other methods above will carry over to the higher-order case as well.

Another trick is *reduction of order*. If you have an equation that only involves derivatives of x (as opposed to x itself), you can set $u = x'$ and you'll get a differential equation for u of smaller order, which ought to be easier to solve.

Be ready for questions about how the gain and phase lag will depend on ω if there is a sinusoidal input. There are examples in notes 16 and recitation notes 11.

Unit 3

Fourier series

The functions $\cos(nt)$ and $\sin(nt)$ have period $2\pi/n$. Fourier series let us write any reasonable function with period 2π as a (possibly infinite) combination of these basic functions:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

Here "reasonable" (not a technical term) means f and its derivative are both piecewise continuous and averaged (the latter of these meaning that at each discontinuity, the function

has $f(a) = (f(a+) - f(a-))/2$. The coefficients can be computed using the integrals

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

This all works for functions of period other than 2π too. If the period is $P = 2L$, the series will have terms $a_n \cos(n(\pi/L)t)$ and $b_n \sin(n(\pi/L)t)$, and there are similar integrals to compute the coefficients.

Actually computing any of these is integrals is a pain, especially on an hour exam. So make as many simplifying observations as you can. If f is odd, then its Fourier expansion will have only sin's in it ($a_n = 0$). If f is even, the expansion has only cos's ($b_n = 0$). The other important strategy is to try to relate our function to one for which the series is already known, by using translations, horizontal rescalings, differentiation, etc. If you know the Fourier series for the original function, you can just perform each of the operations on the Fourier series. The trickiest is dealing with translations: to get the series for $f(t - \pi/6)$ from that for $f(t)$, just substitute to get terms like $3\cos(4(t - \pi/3))$. A term like that isn't allowed in a Fourier series, but you can expand using the sum identity for cosine: $3\cos(4t)\cos(4\pi/3) - 3\sin(4t)\sin(4\pi/3)$, and this is allowed. Expand all the terms that way and you get the Fourier series.

Just about the only function we actually computed the series for was the square wave, which is odd, of period 2π, and takes the value 1 between 0 and π (this data is enough to determine it completely).

$$\mathrm{Sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right)$$

Fourier series turn out to be very useful for differential equations. Here's an "example". If you have a spring system with any periodic external force $q(t)$, it's described by the equation $mx'' + bx' + kx = q(t)$. Expand $q(t)$ as a Fourier series:

$$mx'' + bx' + kx = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots .$$

The terms on the right can be handled one by one: for example $mx'' + bx' + kx = b_2\sin(2t)$ is something you know how to solve using the sinusoidal response method. Find the solution for each term in the Fourier series for $q(t)$ and add them all up. By superposition, this gives a solution to our original equation.

Something to watch out for: one of the terms in the Fourier series might be at the resonant frequency, so you need to pay special attention when handling that term. In this case the solution you find won't be periodic – you'll end up with a term like $t\sin(\omega t)$ for some value of ω.

Delta functions etc.

We think of the delta function as a function that has very large values when t is very close to 0. It's so large, in fact, that $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$. The Heaviside step function $u(t)$ is the function

which is 0 for $t < 0$ and 1 for $t > 0$. Its derivative is the delta function (really all of this should be interpreted in terms of time scales, for which see notes 23). A *regular function* is piecewise smooth, i.e. it's broken into a bunch of pieces and infinitely differentiable on each of them. A *singularity* function is a sum of delta functions. A *generalized function* is the sum of a regular function and a singularity function. To take the derivative of a piecewise function, here's what to do: take the usual derivative of the function of the function everywhere except where the function jumps. If the function jumps by some amount $J = f(a+) - f(a-)$ at $t = a$, then throw in a $J \cdot \delta(t-a)$. To multiply a delta function by a continuous function, don't forget the rule $f(t) \cdot \delta(t-a) = f(a) \cdot \delta(t-a)$. With all these definitions, the product rule and fundamental theorem of calculus both work for generalized functions.

The delta function and unit step function are both useful in setting up differential equations to describe real-life situations. Here are the two basic examples. $x' - kx = q(t)$ describes a bank account, paying interest at rate k, and such that money is deposited at a rate of $q(t)$. If $q(t) = a \cdot u(t)$ is the step function, this means you deposit no money until $t = 0$, and then deposit at a constant rate of 1 for $t > 0$. If $q(t) = a \cdot \delta(t)$, this means we deposit no money for $t < 0$, at $t = 0$ we make a lump-sum deposit of a, and then never deposit money again. In particular the balance is discontinuous and jumps by a at $t = 0$. The second order example is a driven mass-spring system $mx'' + bx' + kx = F_{ext}(t)$. If $F_{ext}(t) = u(t)$, there's no force until $t = 0$, at which time a force of 1 kicks in. If $F_{ext}(t) = \delta(t)$, this means the mass sits until $t = 0$, at which time it is struck very hard. The derivative $x'(0)$ jumps by $1/m$ at $t = 0$, but x is continuous at 0. This is explained in lectures notes 24 (or, in a longer and more rambling form, in a post I made on Piazza). The following table summarizes the initial conditions you need to use.

	First order		Second order	
	Step	Impulse	Step	Impulse
Equation:	$cx' + kx = u(t)$	$cx' + kx = \delta(t)$	$mx'' + bx' + kx = u(t)$	$mx'' + bx' + kx = \delta(t)$
For $t > 0$ solve:	$cx' + kx = 1$	$cx' + kx = 0$	$mx'' + bx' + kx = 1$	$mx'' + bx' + kx = 0$
With conds:	$x(0) = 0$	$x(0) = 1/c$	$x(0) = 0$, $x'(0) = 0$	$x(0) = 0$, $x'(0) = 1/m$
Don't forget to multiply by $u(t)$ at the end!				

The unit step response for an operator $p(D)$ is the function $x(t)$ that satsifies $p(D)x = u(t)$, starting with initial conditions $x^{(n)}(0-) = 0$. The unit step response is a solution to $p(D)x = 1$, with all initial conditions 0 (for first order, this means $x(t) = 0$, for second order, $x(t) = x'(t) = 0$). You know how to solve $p(D)x = 1$ from last unit: solve the homogeneous equation, find a particular solution, add them). Don't forget to multiply your solution by $u(t)$, since you want it to be 0 for $t < 0$.

The unit impulse response for an operator $p(D)$ solves $p(D)x = \delta(t)$. For $t > 0$ this is a solution to the homogeneous equation $p(D)x = 0$ with initial conditions all 0, except the highest-order derivative has $x^{(n)}(0) = 1/a_n$, where a_n is the leading coefficient of the operator (for the bank example: $x(t) = 1$; for the spring example: $x(t) = 0$, $x'(t) = 1/m$). Often the unit impulse response is denoted $w(t)$. You can also find the step and impulse response using Laplace transform. More on this later.

Given two functions $f(t)$ and $g(t)$, the convolution is defined by the integral

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

This is another function of t. The official 18.03 real-life example of convolution is something about farm runoff, for which I refer you to lecture notes 25. Key fact: convolution is commutative, so $f(t) * g(t) = g(t) * f(t)$. The reason we care about this is that if $w(t)$ is the weight function above, then a solution to $p(D)x = q(t)$ with rest initial conditions is given by $q(t) * w(t)$. This means that if you know the weight function for an operator, you can get a solution to $p(D)x = q(t)$ by computing the convolution integral (however in most cases doing this integral will be harder than just solving the equation directly). Keep in mind that if are you trying to convolve two functions, it's often good to take "g" to be the simpler one, so you'll have fewer τ terms involved in the integral.

Laplace transform

Given a function $f(t)$ (satisfying some conditions that I won't worry about), the Laplace transform is a function $F(s)$, defined by $\mathcal{L}[f] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$. It satisfies a bunch of useful relations. No need to memorize these: you'll get a table on the exam.

Linearity	$\mathcal{L}[a\,f(t) + b\,g(t)] = a\,F(s) + b\,G(s)$	$\mathcal{L}[1] = \frac{1}{s}$
s-shift	$\mathcal{L}[e^{rt}f(t)] = F(s-r)$	$\mathcal{L}[\delta(t-a)] = e^{-a}$
t-shift	$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{at}] = 1/(s-a)$
s-derivative	$\mathcal{L}[tf(t)] = -F'(s)$	$\mathcal{L}[t^n] = n!/s^{n+1}$
t-derivative	$\mathcal{L}[f'(t)] = s\,F(s) - f(0-)$	$\mathcal{L}[\cos(\omega t)] = s/(s^2+\omega^2)$
	$\mathcal{L}[f''(t)] = s^2F(s) - s\,f(0-) - f'(0-)$	$\mathcal{L}[\sin(\omega t)] = \omega/(s^2+\omega^2)$
Convolution	$\mathcal{L}[f(t) * g(t)] = F(s)\,G(s)$	$\mathcal{L}[t\,\cos(\omega t)] = 2\omega s/(s^2+\omega^2)^2$
Weight fct	$\mathcal{L}[w(t)] = W(s)$	$\mathcal{L}[t\,\sin(\omega t)] = (s^2-\omega^2)/(s^2+\omega^2)^2$

The significance of this is that given a differential equation, you can hit both sides with the Laplace transform and it turns into an algebraic problem. Note that taking the transform of f' and f'' is where the initial conditions come into play: the transform of derivatives of f depends on $f(0-)$ $f'(0-)$, ... by the t-derivative rule. (NB: pre-initial conditions, as opposed to the post-initial in the earlier method for impulse response) You can then solve for the Laplace transform $F(s)$ of the solution, and then try to find a function having the given transform. There is of course no free lunch, and if you use this method the hard part is typically finding the inverse transform of $F(s)$.

To find the impulse response $p(D)w = \delta(t)$, taking the transform yields $p(s)W(s) = 1$, so $p(s) = 1/W(s)$. This $W(s)$ is called the transfer function. If you know the transfer function, you can find the weight function by taking inverse transform. Laplace transform also lets you find the operator given the weight function: compute the transform of $w(t)$, find its reciprocal $1/W(s) = p(s)$, and the result is the characteristic function of the operator!

Often it's useful to draw the "pole diagram" of $W(s)$, or more generally any Laplace transform $F(s)$. This just means to draw the complex plane and put a dot at every point where $W(s)$ is infinite (usually $W(s)$ is a rational function, so the dots go at the zeroes of

the polynomials in the denominator). The pole diagram of a function $F(s)$ tells you quite a bit about the behavior of $f(t)$. Very roughly, a pole at $a + bi$ tells you your function has a term that acts like $e^{at}\cos(bt)$. For large t, all of these summands will be negligible except the one for which the real part a is largest, since the other exponentials decay more quickly. So the long term behavior is governed by by pole that is the rightmost in the diagram. If all poles have negative real part, $f(t)$ decays to 0, ...

Linear Algebra

I won't give a detailed treatment of all our linear algebra. Here's the important vocabulary. The source of much confusion is that many of these gadgets can be interpreted both purely algebraically, or as describing something geometric, and we often switch between the two viewpoints.

Term	Algebraic	Geometric
Vector	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	Point in $\mathbb{R}^3$
Matrix	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	Linear transformation
Invertible	There exists a matrix A^{-1} with $AA^{-1} = I$	some other linear transformation reverses it (unlike e.g. projection to x-axis)
Matrix . vector	Matrix mult.	Image of v under linear transformation
Matrix mult.	Know how to do it!	Composition of linear transformations
Subspace	Set of vectors closed under additon	Line in a plane/plane in space etc.
Span	All vectors $av_1 + bv_2$	Plane spanned by v_1 and v_2
Determinant	Sum of downward diags - sum of upward (for dim $= 2, 3$ only!)	Volume of parallelpiped spanned by vectors
Linear indep.	$av_1 + bv_2 + cv_3 = 0$ implies $a = b = c = 0$	No two vectors parallel, no three in one plane
Linear dep.	Some linear combo is 0	Three coplanar vectors, or similar
Basis	Set of vects such that any vector can be expressed as combo $av_1 + bv_2 + cv_3$	Two non-parallel vectors in a plane, etc.
Char. poly	$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + (\det)A$	Not as nice – Google it if curious
Eigenvalue	Zeroes of char poly	Scaling factor for eigenvector
Eigenvector	Vector with $A\vec{v} = \lambda\vec{v}$	Vector that points in same direction after linear trans

Many of these are related. To check if three vectors in $\mathbb{R}^3$ are linearly independent, put the three in a matrix, and compute the determinant. It's 0 if and only if the vectors are

linearly *de*pendent. If one of your vectors depends on a parameter a, you can find the a which makes the vectors dependent by setting the determinant equal to 0 and solving. A set of three vectors in $\mathbb{R}^3$ forms a basis if and only if they are linearly independent.

The eigenvalues of a matrix M are exactly the zeroes of the characteristic polynomial $p_M(t)$. To find an eigenvector for an eigenvalue λ, write down the matrix $A - \lambda I$, and find a $\vec{v}$ which is sent to 0 by this matrix. In the 2×2 case, you can just swap the entries of the top row of $A - \lambda I$ and add a minus sign to one of them. Also worth rememberingis that the sum of the eigenvalues is equal to the trace (sum of diagonal entries), and the product of the eigenvalues is the determinant.

We're interested in differential equations of the form $\mathbf{u}(t) = A\mathbf{u}(t)$, where $\mathbf{u}(t) = (x(t), y(t))$ and A is a matrix (usually 2×2 in this course, lest the algebra get out of hand). This means the derivatives of $x(t)$ and $y(t)$ depend on both $x(t)$ and $y(t)$. If you're so inclined, you can turn a second-order homogeneous linear equation $\ddot{x} + b\dot{x} + kx = 0$ into a first order linear system. Let $y = \dot{x}$. Then $\left(\begin{smallmatrix} \dot{x} \\ \dot{y} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -k & -b \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. The matrix appearing here is called the *companion matrix*.

As with 2nd order equations, if you can find two basic solutions $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, then every other solution will be a linear combination of the two. Given initial conditions $x(0)$ and $y(0)$, you can pick constants c and d so $c\mathbf{u}_1(t) + d\mathbf{u}_2(t)$ has the initial conditions you're after.

So the challenge is to find a couple particular solutions. The easiest to find are *ray solutions*, when $(x(t), y(t))$ just moves along a single line. How can this happen? Suppose you have a solution on the ray spanned by $\mathbf{u}_1$, so it is of the form $\mathbf{u}(t) = f(t)\mathbf{u}_1$. Then $\mathbf{u}'(t) = f'(t)u_1$, so we want $f'(t)\mathbf{u}_1 = f(t)A\mathbf{u}_1$. This is an equality of vectors – for this to happen, the two vectors have to be in the same direction, and so $\lambda_1\mathbf{u}_1 = A\mathbf{u}_1$ for some value of λ. That is to say, $\mathbf{u}_1$ is an eigenvector of A! Then $f'(t) = \lambda_1 f(t)$, which is solved by $e^{\lambda_1 t}$.

If you can find two distinct real eigenvectors, you get two solutions, and you plug in the initial conditions and everything is under control. But some matrices have complex eigenvectors, and cases with repeated or 0 eigenvalues can pose special problems. If you have two complex conjugate eigenvectors, here's the plan: pick a complex eigenvalue, find an eigenvalue λ_1 with eigenvector $\mathbf{v}_1$ (both of which will be complex). A solution is again $e^{\lambda_1}\mathbf{v}_1$. The real and imaginary parts of this product give two independent solutions. In general, this will be a product of an exponential and a trig function, like we've encountered before.

The other problem is if there's only one real root. Two things can happen here. If the matrix is diagonal, then every vector is an eigenvector, and a solution with initial conditions $\mathbf{u}(0) = \mathbf{v}_0$ is given by $e^{\lambda t}\mathbf{v}_0$. If it's not diagonal, you can find one eigenvector $\mathbf{v}$, and another vector with $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Your solutions are then $c_1 e^{\lambda_1 t}\mathbf{v} + c_2 e^{\lambda_1 t}(t\mathbf{v} + \mathbf{w})$. This is the so-called *defective* case.

That covers all the main cases. There's another way to think about when these arise. Since $\operatorname{tr} A = \lambda_1 + \lambda_2$ and $\det A = \lambda_1\lambda_2$, in principle you can solve for the eigenvalues given the trace and determinant. So it should come as no surprise that the relationship between trace and determinant tells us what sorts of eigenvalues we get. You can plot the two against each other, and the critical curve is $\det(A) = \operatorname{tr}(A)^2/4$. If A lies above this curve, it has two complex eigenvalues. If it lies between $\det = 0$ and $\det = \operatorname{tr}(A)^2/4$, it has two real eigenvalues of the same sign. If it lies below the curve, it has two real eigenvalues of opposite sign.

The behavior of the solutions to $\dot{\mathbf{u}} = A\mathbf{u}$ depends on the types of the eigenvalues of $\mathbf{u}$. It's extremely helpful to draw the *phase portrait* of such a system. This shows the xy-plane, with curves drawn for a few solutions, illustrating $x(t)$ and $y(t)$ vary as t changes. I'm not going to include the pictures here, but definitely take a look at supplemental notes 31 to see what's going on. The various possibilities

Eigenvalues	det/tr	type	example
$\lambda_1 < 0, \lambda_2 > 0$	$\det A < 0$	Saddle	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
$\lambda_1 < 0, \lambda_2 < 0$	$0 < \det A < (\operatorname{tr} A)^2/4$, $\operatorname{tr} A < 0$	Stable node	$\left(\begin{smallmatrix} -1 & 0 \\ 0 & -2 \end{smallmatrix}\right)$
$\lambda_1 > 0, \lambda_2 > 0$	$0 < \det A < (\operatorname{tr} A)^2/4$, $\operatorname{tr} A > 0$	Unstable node	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$
$\lambda_i = a \pm bi$, $a < 0$	$\det A > (\operatorname{tr} A)^2/4$, $\operatorname{tr} A < 0$	Stable spiral	$\left(\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$
$\lambda_i = a \pm bi$, $a > 0$	$\det A > (\operatorname{tr} A)^2/4$, $\operatorname{tr} A > 0$	Unstable spiral	
$\lambda = \pm bi$	$\det A > 0$, $\operatorname{tr} A = 0$	Center	$\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$
$\lambda_1 = \lambda_2 < 0$	$\det A = (\operatorname{tr} A)^2/4$	Stable defective node (not diagonal)	$\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$
		Stable star (diagonal)	$\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
$\lambda_1 = \lambda_2 > 0$	$\det A = (\operatorname{tr} A)^2/4$	Unstable defective node (not diagonal)	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$
		Unstable star (diagonal)	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
$\lambda_1 = 0, \lambda_2 < 0$	$\det A = 0$, $\operatorname{tr} A < 0$	Stable comb	$\left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
$\lambda_1 = 0, \lambda_2 < 0$	$\det A = 0$, $\operatorname{tr} A < 0$	Unstable comb	$\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
$\lambda_1 = \lambda_2 = 0$	$\det A = \operatorname{tr} A = 0$	Parallel lines (not diagonal)	$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$
		Constant (diagonal)	$\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$

Here "stable" means that solutions go to $(0, 0)$ as t goes to infinity, while unstable means some solution goes to infinity. Stability is determined by the signs of the real parts of the eigenvalues: an eigenvalue with positive real part indicatives an unstable system.

Note that the eigenvalues alone don't tell you everything. A star and a defective node both have a single repeated real eigenvalue. To distinguish these cases, you need to check whether A is diagonal.

Another important feature that you can't see from the eigenvalues alone is whether a spiral will be clockwise or counterclockwise. There's an easy way to check this: plug in $(1, 0)$ for $\mathbf{u}$. That gives you $\dot{\mathbf{u}}$ at this point. If it points upwards, the spiral is counterclockwise, and vice versa.

Make sure you have some idea how to draw these! Here's the gist for the basic three. For a saddle, find the eigenvectors v_1 and v_2. Say v_1 has a negative eigenvalue, and v_2 a positive eigenvalue. There is an incoming ray solution along v_1 and an outgoing ray solution along v_2. Draw these two ray solutions, and then the other solutions interpolate between these two. For a node, again find the two eigenvectors. You get either two incoming or two outgoing ray solutions; draw both of these. Then draw in the other solutions (all incoming, and becoming closer to the one with the eigenvalue of smaller absolute value). For a spiral, figure out if it's going clockwise or counterclockwise, and maybe check the direction at a few specific points.

By now we know how to find two solutions to $\dot{\mathbf{u}} = A\mathbf{u}$. If you put two (or n, if your system has more equations) of them next to each other as the columns of a matrix, you get a *fundamental matrix* Φ. A general solution to the differential equation is then $c_1u_1 + c_2u_2 = \Phi\mathbf{c}$, where $c = \binom{c_1}{c_2}$. This isn't unique – it depends on your choice of solutions. There is a special one, though, the exponential matrix e^{At}. It arises when you choose normalized solutions with $\mathbf{u}_1(0) = (1, 0)$, $\mathbf{u}_2(0) = (0, 1)$. If you have another fundamental matrix, you can use $e^{At} = \Phi(t)\Phi(0)^{-1}$. This gives a nice shortcut: the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with $\mathbf{u}(0) = \mathbf{u}_0$ is given by $\mathbf{u}(t) = e^{At}\mathbf{u}_0$. (This may sound profound, but it is really just a reformulation in terms of matrices of our usual procedure for finding constants using initial conditions. Nonetheless it can save you some algebra).

We're often interested in inhomogeneous systems $\dot{\mathbf{u}} = A\mathbf{u} + F(t)$. As in the second-order case, the general solution is $\mathbf{u}_h + \mathbf{u}_p$, where $\mathbf{u}_h$ is the general solution to the homogeneous equation $\dot{\mathbf{u}} = A\mathbf{u}$, and $\mathbf{u}_p$ is a single solution to $\dot{\mathbf{u}} = A\mathbf{u} + F(t)$. If $F(t) = F$ is constant, you can just use $\mathbf{u}_p = -A^{-1}F$.

Many important differential equations aren't linear, but these methods are also useful in studying nonlinear autonomous sytems, in which x' and y' depend on x and y, but not on t: $x'(t) = F(x, y), y'(t) = G(x, y)$. You can draw a vector field for the trajectories to help visualize them. The critical points are where x' and y' are both 0. To find them, you can just solve for when F and G are both 0.

Near an equilibrium, the system is well-approximated by the linear equation $\dot{u} = Ju$, where $J = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}$. So you can study the behavior near the critical points using the linear approximation and our study of the linear versions. Here are the steps:

1. Find the critical points, by solving for points where $\dot{x} = 0$ and $\dot{y} = 0$.

2. At each one, compute the Jacobian at that point.

3. Analyze the linear equation $\dot{\mathbf{u}} = J\mathbf{u}$ as above (e.g. figure out how to draw it)

How to interpret this? The linearization gives a good approximation of the phase portrait if we're close to the critical point in question. To sketch the full phase portrait, draw in the spirals, nodes, etc. you found at the critical points – this is where the interesting action is. Elsewhere, it extrapolates between what's going on at these critical points. Plugging in a few values can help you draw the function elsewhere. Once you've done this, you should be able to eyeball the sketch and see the asymptotic behavior of many solutions (e.g. starting at $(1, 0)$, it will eventually converge to the critical point $(2, 2)$, which is a stable spiral with angular pseudofrequency...).