Exam II review

If any of these solutions are indecipherable or outright wrong, let me know and I'll post a correction! If I skipped a step and you want more details, email me.

1. Expand $3\cos(2t-\pi/3)$ as $a\cos(2t)+b\sin(2t)$. Write $\cos(3t)-\sin(3t)$ in standard form.

   It's $3\cos(2t)\cos(\pi/3)+3\sin(2t)\sin(\pi/3)=\frac{3}{2}\cos(2t)+\frac{3\sqrt{3}}{2}\sin(2t)$.

   For the second, we have $A=\sqrt{a^2+b^2}=\sqrt{2}$. Then $\cos\phi=a/A=1/\sqrt{2}$ and $\sin\phi=b/A=-1/\sqrt{2}$, so $\phi=-\pi/4$. The answer is then $\sqrt{2}\cos(3t+\pi/4)$.

2. Consider the equation $x''+bx'+9x=0$.

   - For what values of b is it underdamped? Overdamped? Critically damped?
     The characteristic polynomial is $p(s)=s^2+bs+9$, and the roots are $\frac{-b\pm\sqrt{b^2-36}}{2}$. The are two real roots if $b^2-36>0$, meaning $b>6$ (or $b<-6$, which doesn't make sense if this is a frictional force). This is the overdamped case. There are two conjugate complex roots if $b<6$, the underdamped case. There's a single real root if $b=6$; this is critical damping.
   - Suppose we observe that $x(\pi/4)$ and $x(3\pi/4)$ are both zero. Which of the above cases must we be in? What must be the value of b?

     Overdamping and critical damping both have a single zero. So it must be that it is underdamped. The solution is $Ce^{at}\cos(\omega t-\phi)$. ω here will be the imaginary part of the root of the characteristic polynomial, namely $\sqrt{36-b^2}/2$. For $\cos(\omega t)$ to have zeroes that are spaced by $\pi/2$, we need $\omega=2$. Thus $\sqrt{36-b^2}/2=2$ and $b=2\sqrt{5}$.

3. What is the general solution to

   $$x''+3x'+2x=4e^t+e^{-2t}+(12t^2+14t+14)e^{2t}+t^2+e^t\cos(t)+e^t\cos(t+\pi/4)$$

   It's not as bad as it looks. By the principle of superposition, we can just find a solution to a solution to each of the right hand terms separately and then add them up. So really this is just seven or so problems stuck together.

   The general solution we find by solving the characteristic polynomial $p(s)=s^2+3s+2$. The roots are -1 and -2, so the general solution is

   $$x_g=c_1e^{-t}+c_2e^{-2t}$$

   Now find a particular solution for $4e^t$. This isn't resonant, so ERF will work. We get $4/p(1)e^t=\frac{2}{3}e^t$.

   Next up is e^{-2t}. Note that $p(-2)=0$, so this is resonant. ERF for resonance gives $1/p'(-2)te^{-2t}=-te^{-2t}$.

How about e^{2t}? This is one for variation of parameters. Suppose the solution is $u(t)e^{2t}$. Then $x' = u'(t)e^{2t} + 2u(t)e^{2t}$ and

$$\begin{aligned} x'' &= u''(t)e^{2t} + 2u'(t)e^{2t} + 2u'(t)e^{2t} + 4u(t)e^{2t} \\ &= \left(u''(t) + 4u'(t) + 4u(t)\right) e^{2t} \end{aligned}$$

This means

$$x'' + 3x' + 2x = \left((u'' + 4u' + 4u) + 3(u' + 2u) + 2u\right) e^{2t} = (u'' + 7u' + 12u)e^{2t}.$$

We'll be in business if $u'' + 7u' + 12u = 12t^2 + 14t + 14$. Here we use the method of unknown coefficients. Certainly u can't be a polynomial of degree more than 2. Set $u = at^2 + bt + c$, expand everything, and solve for a, b, c and I think you'll get $u = t^2 + 1$. So the solution is $(t^2 + 1)e^{2t}$.

The t^2 is easy to deal with. Set $x(t) = at^2 + bt + c$. Then $x'' + 3x' + 2x = 2at^2 + (6a + 2b)t + (2a + 3b + 2c)$, whence $a = 1/2$, $b = -3/2$, $c = 7/4$. So the solution is $\frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{2}$.

For $e^t \cos(t)$ we use complex replacement and ERF. This gives $z'' + 3z' + 2z = e^{(1+i)t}$. ERF says that we can use $1/p(1 + i) = 1/(5 + 5i)$. So the answer is the real part of $\frac{1}{5+5i}e^{(1+i)t} = \frac{1}{5\sqrt{2}}e^{-i\pi/4}e^{(1+i)t} = \frac{1}{5\sqrt{2}}e^t e^{(t-\pi/4)i}$, which is $\frac{\sqrt{2}}{10}e^t \cos(t - \pi/4)$.

Now $e^t \cos(t + \pi/4) = e^{-\pi/4}e^{t+\pi/4} \cos(t + \pi/4)$. Here we use time shifting and linearity. So we can take our solution to the previous one, multiply it by $e^{-\pi/4}$, and plug in $t + \pi/4$ everywhere we see a t, obtaining a particular solution

$$\frac{e^{-\pi/4}\sqrt{2}}{10}e^{t+\pi/4} \cos t = \frac{\sqrt{2}}{10}e^t \cos t.$$

Our final solution is

$$f(x) = c_1e^{-t} + c_2e^{-2t} + \frac{2}{3}e^t + (t^2 + 1)e^{2t} - te^{-2t} + \frac{\sqrt{2}}{10}e^t \cos(t - \pi/4) + \frac{\sqrt{2}}{10}e^t \cos t.$$

Phew!

4. Think about the equation $x'' + 2x' + kx = \cos(2t)$. How does the gain change as k incrseases? The phase lag? For what value of k is there a $\pi/2$ phase lag? $\pi/4$? (Note: I changed the sign here to make the problem work. Sorry!)

   We use the ERF. The solution will be $1/p(2)e^{2it} = 1/((k-4)+2i) \cdot e^{2it}$. As k increases, the gain goes to 0, so the solutions are very small. Similarly the argument goes to 0, and the phase lag does too.

   To get a phase lag of $\pi/2$, we want the negative of the argument of $H(\omega)$ to be $\pi/2$, so $H(\omega)$ should be a negative real number times i. This means $(k - 4) + 4i$ should be a positive real number times i, which will happen if $k = 4$. To get a phase lag of $\pi/4$, we need $(k - 4) + 4i$ to have argument $\pi/4$, meaning it has real part equal to imaginary part. This happens when $k = 8$.