Exam III review

If any of these solutions are indecipherable or outright wrong, let me know and I'll post a correction! If I skipped a step and you want more details, email me.

1. Let $f(t)$ be the function of period 2 with $f(t) = 1$ for $0 \le t < 1$ and $f(t) = 0$ for $1 \le t < 2$.

   (a) Compute the Fourier series for f.

   (b) How would you solve the differential equation $x' + 3x = f(t)$ using Fourier series?

   The function is $\frac{1}{2}sq(\pi t) + \frac{1}{2}$. Since $sq(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right)$. So

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(\pi t) + \frac{\sin 3\pi t}{3} + \frac{\sin 5\pi t}{5} + \cdots\right)$$

   Solve it using superposition and exponential response. $x' + 3x = \frac{\sin k\pi t}{k}$ can be solved by taking the imaginary part of $z' + 3z = \frac{e^{ik\pi t}}{k}$. By exponential response, that's the imaginary part of $\frac{1}{k}\frac{3-ik\pi}{9+k^2\pi^2}e^{ik\pi t}$.

$$x_p = \frac{1}{k(9+k^2\pi^2)}(3\sin k\pi t - k\pi\cos k\pi t)$$

   We also need to deal with the 1/2 term, which is $x = 1/6$. So

$$x(t) = \frac{1}{6} + \sum_{k \text{ odd}} \frac{1}{k(9+k^2\pi^2)}(3\sin k\pi t - k\pi\cos k\pi t).$$

2. Consider the function

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ t^2 & \text{for } 0 \le t \le 1 \\ t-1 & \text{for } t > 1 \end{cases}$$

   (a) Sketch a graph of $f(t)$ and $f'(t)$. Is $f(t)$ piecewise continuous? Averaged?
   I'm too lazy to enter a graph. f' is 0 for $t < 0$, $2t$ for $0 \le t \le 1$, and 1 for $t > 1$. There's no jump at 0, but there is at 1, so you should have a downward-pointing harpoon of magnitude 1 at $t = 1$. It's not averaged – for that, you need $f(1) = 1/2$.

   (b) Express $f(t)$ in terms of the unit step function $u(t)$, and compute $f'(t)$ algebraically. Do you get the same answer?
   We can write

$$f(t) = u(t)t^2 + u(t-1)(t-t^2).$$

Then

$$\begin{aligned} f'(t) &= 2t\,u(t) + t^2\,\delta(t) + (1-2t)u(t-1) + (1-2t)\,\delta(t-1) \\ &= 2t\,u(t) + (1-2t)u(t-1) - \delta(t-1) \end{aligned}$$

Work out what this is on each interval, and you'll see that it matches the version we got graphically.

3. (a) Consider the operator $D^2+3D+2I$. Compute the unit impulse response directly, and using Laplace transform.

   For the direct method, we want a solution to $x''+3x'+2x=0$ with initial conditions $x(0)=0$, $x'(0)=1$. The general solution is $c_1e^{-t}+c_2e^{-2t}$, and we need $c_1+c_2=0$ and $-c_1-2c_2=1$, so $c_1=1$, $c_2=-1$. So $w(t)=e^{-t}-e^{-2t}$.

   For Laplace, taking the transform of both sides of $x''+3x'+2x=0$ gives $s^2X(s)+3sW(s)+2W(s)=1$, $X(s)=1/(s^2+3s+2)$. To invert, expand by partial fractions

$$\frac{1}{(s+1)(s+2)} = \frac{a}{s+1} + \frac{b}{s+2}.$$

   Covering up, $a=1$, $b=-1$, so $X(s)=1/(s+1)-1/(s+2)$. This has inverse transform $e^{-t}-e^{-2t}$, the same thing we got the other way!

   (b) How would you use this to solve $x''+3x'+2x=\cos t$ with rest initial conditions? Set up the relevant convolution integral.

   The solution is

$$w(t)*\cos t = \int_0^t (e^{-\tau}-e^{-2\tau})\cos(t-\tau)\,d\tau.$$

   (c) Find the Laplace transform of the solution to $x''+3x'+2x=\cos t$, with initial conditions $x(0)=0$, $x'(0)=1$.

   Take the transform of both sides:

$$\begin{aligned} \frac{s}{s^2+1} &= s^2X(s) - sx(0-) - x'(0-) + 3sX(s) - 3x(0-) + 2X(s) \\ &= s^2X(s) - 1 + 3sX(s) + 2X(s) = (s^2+3s+2)X(s) - 1. \end{aligned}$$

   So

$$X(s) = \frac{1}{s^2+3s+2}\left(1 + \frac{s}{s^2+1}\right).$$

   Now you have to invert the transform. First step is to expand this mess by partial fractions.

4. Let $f(t)=\cos(2t)+e^{-t}\sin(3t)$. Compute $F(s)=\mathcal{L}[f]$ and sketch the pole diagram.

   $F(s)=\frac{s}{s^2+4}+\frac{3}{(s+1)^2+9}$. The poles are at $2i, -2i, -1+3i, -1-3i$. The ones with the largest real part are $2i$ and $-2i$, so in the long term it should look like $\cos 2t$ or $\sin 2t$. Which it does, since e^{-t} is awfully small when t is large!

5. Compute the inverse Laplace transforms of $F_1(s) = 1/(s^2(s+1))$, $F_2(s) = e^{-s}/s^2$.

Expand by partial fractions

$$\frac{1}{s^2(s+1)} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s+1}.$$

By coverup, $a = 1$, $c = 1$. You need another way to get b. One option is to plug in $s = 1$, whence $1/2 = 1 + b + 1/2$, so $b = -1$. This means

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}.$$

The inverse transform is $t - 1 + e^{-t}$.

For e^{-s}/s^2, two steps. The inverse transform of $1/s^2$ is $f(t) = t$, and now we want to use the t-shift formula. That requires that the function we are shifting is 0 for $t < 0$, which t isn't. However, $f(t) = t \cdot u(t)$ is 0 for $t < 0$, and it still has transform $1/s^2$, so we can apply the t-shift rule to that. Thus the correct inverse transform is $(t-1) \cdot u(t-1)$.