Disclaimer: I haven't checked all of the solutions too carefully. If something seems fishy it probably is – send me an email (`johnl@math.mit.edu`) and I'll have a look.

1. Can you come up with a matrix A such that the system $\dot{\mathbf{u}} = A\mathbf{u}$ has a saddle, with an incoming ray solution along $(1, 1)$ and an outgoing one along $(1, 2)$?

   We want
   $$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}, \qquad A\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$
   (This takes the eigenvalues to be -1 and 1 – you could use others if you want, as long as the signs are the same). Putting these together,
   $$A\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}-1 & 1\\-1 & 2\end{pmatrix},$$
   So $A = \begin{pmatrix}-1 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1} = \begin{pmatrix}-3 & 2\\-4 & 3\end{pmatrix}$ will do the trick.

2. Find the general solution to $y' + (\sin x)y = e^{\cos x}$.

   It's been awhile since we did one of these. Solve the homogenous equation $y' + (\sin x)y = 0$. This is separable, and we have $dy/y = -\sin x\, dx$ whence $y = e^{\cos x}$ is a solution. Now use variation of parameters: assume $y = u\, e^{\cos x}$. Then
   $$y' + (\sin x)y = u(e^{\cos x})(-\sin x) + \sin x u e^{\cos x} = u' e^{\cos x}.$$
   We want this to be equal to $e^{\cos x}$. How convenient! This just means $u' = 1$, so $u = x + c$, and the general solution is $y(x) = (x + c)e^{\cos x}$. This example worked out neatly, and it's always pretty easy to see what to do when using variation of parameters. The catch is that you might need u' to be equal to something you don't know how to integrate. (Not that this is your fault! The functions that appear might not have any closed-form antiderivative).

3. (1.3.6) Consider the differential equation $\frac{dy}{dx} = x-y+1$. Sketch some isoclines and describe the behavior of the integral curves. Which solutions have local maxima or minima? Points of inflection? Can you identify any fences? Estimate $y(10)$ for the solution with $y(0) = 2$.

   The plot is in the book. Every solution seems to converge to the line $y = x$. This means that $y(10)$ is probably about 10 as long as we didn't start too far away from the line $y = x$. Minima occur where $y' = 0$, which is the nullcline $y = x + 1$. Inflection points have $y'' = 0$, so $1 - y' = 0$. These occur on the $m = 1$ isocline, which is itself a solution. So no other solutions have inflection points.

I claim that every isocline is eventually a fence, which explains this observation. For concreteness, let's consider the $m = 2$ isocline. It's defined by $2 = x - y + 1$, i.e. $y = x - 1$. Above the isocline, but below $y = x$, things have slope between 1 and 2. So they can't cross back below the isocline itself, since it has slope 1.

This example doesn't really show the trickier things that can happen with fences. In general the thing to do is find the equation for the isocline and take its derivative; you'll get a fence when the slope of the m-isocline is always greater than (or always less) than m in some range. Check out the second recitation solutions for a harder example.

4. Look at $\ddot{x} + 3\dot{x} + kx = k\cos(3t)$. What is the complex gain (regarding $\cos 3t$ as the input signal)? For what value of k is the amplitude of the system response maximized? How does the phase lag vary with k?

   Complex replacement gives $\ddot{z} + 3\dot{z} + kz = ke^{3it}$. Here $y_{cx} = e^{3it}$ is the complexified input signal. By ERF, the solution is $z = \frac{k}{(k-9)+(9i)}e^{3it}$. The complex gain is the factor $\frac{k}{(k-9)+9i}$.

   The amplitude is largest when $\left|\frac{k}{(k-9)+(9i)}\right|^2 = \frac{k^2}{(k-9)^2+81}$ is maximized. This happens for $k = 18$, as you can see by doing some calculus.

5. Let $f(t) = t \cdot |t|$. Express this function in terms of $u(t)$. What is $f'(t)$? $f''(t)$? $f'''(t)$? Sketch the last of these.

   Note that $|t|$ is nothing but $(2u(t)-1)t$. So our function is $(2u(t)-1)t^2 = 2t^2u(t)-t^2$. By the product rule, $f'(t) = 4tu(t)+2t^2\delta(t)-2t = 4tu(t)-2t$ (this is just $2|t|$!). The second derivative is $f''(t) = 4t\delta(t) + 4u(t) - 2 = 4u(t) - 2$, a step function which jumps from -2 to $+2$ at 0. The third derivative is $4\delta(t)$. There's not much to sketch!

6. Consider the autonomous equation $\frac{dy}{dt} = (y - c)(y^2 - c)$, where c is a real number. What are the equilibrium solutions? Are they stable? (The answers to these questions depend on c). Sketch the bifurcation diagram.

   The equilibria are at c, $\sqrt{c}$, and $-\sqrt{c}$ (the latter two existing only if $c \geq 0$.

If $c < 0$ there is a single equilibrium, which is unstable. If c is between 0 and 1, then $-\sqrt{c} < c < \sqrt{c}$, and the points are unstable, stable, unstable in that order. If $c = 1$, there are only two equilibria, of which -1 is unstable and 1 is semistable. If $c > 1$ then $-\sqrt{c} < \sqrt{c} < c$ and these are unstable, stable, unstable.

7. Compute $\int e^t \cos t \, dt$.

   Complex replacement: this is

$$Re \int e^{(1+i)t}\, dt = Re\left(\frac{1}{1+i} e(1+i)t\right) = Re\left(\frac{1-i}{2} e^{(1+i)t}\right) = \frac{e^t \cos t}{2} + \frac{e^t \sin t}{2}.$$

8. Let A be the 2×2 matrix $\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$. Compute the determinant and trace, and describe type of the phase portrait for different values of a. What curve in the tr-det plane is traced out as a varies?

   $\det A = a^2 - 1$ and $trA = 2a$. Thus $\det(A) = (trA/2)^2 - 1$. This is parallel to the critical curve but lies underneath it, so it never crosses. However, it meets the tr axis at -1 and 1, and the det axis at 0. So: $a < -1$, it's a stable node. $a = -1$, it's a stable comb. $-1 < a < 1$ it's a saddle, and $a = 1$ unstable comb, and $a > 1$ an unstable node. Note that crossing the vertical axis doesn't change anything – it's saddles everywhere below horizontal axis, whether the trace is 0 or not.

9. Find the unit impulse response for $D^2 + 7D + 12I$:

   (a) By solving $\ddot{x} + 7\dot{x} + 12x = 0$ with appropriate initial conditions.

   The right (post-)initial conditions are $x(0) = 0$ and $x'(0) = 1$. The general solution is $c_1 e^{-3t} + c_2 e^{-4t}$, and we need $c_1 + c_2 = 0$ and $-3c_1 - 4c_2 = 1$; clearly $c_1 = 1$, $c_2 = -1$ is going to do the trick:

$$w(t) = e^{-3t} - e^{-4t}.$$

   (b) Using Laplace transform.

   We want $\ddot{x} + 7\dot{x} + 12x = \delta$. Taking the transform of both sides, $s^2W + 7sW + 12W = 1$, so $X = 1/(s^2+7s+12) = 1/(s+3) - 1/(s+4)$ (NB: I am using the fact that the initial conditions are 0, or there would be more terms on the lift side!). Inverting the transform using the s-shift rule gives $w(t) = e^{-3t} - e^{-4t}$.

(c) By setting up the companion matrix and solving the 2×2 system using matrix exponential together with initial conditions.

(This method is probably slightly silly if all you're after is the impulse response)

Let $y = \dot{x}$. We want

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -12 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic polynomial is $p_A(\lambda) = \lambda^2 + 7\lambda + 12 = (\lambda+3)(\lambda+4)$. A (-3)-eigenvector is $\left(\begin{smallmatrix} 1 \\ -3 \end{smallmatrix}\right)$. A (-4)-eigenvector is $\left(\begin{smallmatrix} 1 \\ -4 \end{smallmatrix}\right)$. (These are what we expect; think about it)

So a fundamental matrix is

$$\Phi(t) = \begin{pmatrix} e^{-3t} & e^{-4t} \\ -3e^{-3t} & -4e^{-4t} \end{pmatrix}$$

Then $\Phi(0) = \left(\begin{smallmatrix} 1 & 1 \\ -3 & -4 \end{smallmatrix}\right)$ and $\Phi(0)^{-1} = \left(\begin{smallmatrix} 4 & 1 \\ -3 & -1 \end{smallmatrix}\right)$, so

$$e^{At} = \begin{pmatrix} e^{-3t} & e^{-4t} \\ -3e^{-3t} & -4e^{-4t} \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t} \end{pmatrix}$$

We want $\left(\begin{smallmatrix} x(0) \\ y(0) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, so

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= e^{At} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-3t} - e^{-4t} \\ -3e^{-3t} + 4e^{-4t} \end{pmatrix} \end{aligned}$$

In particular this gives $x(t) = e^{-3t} - e^{-4t}$, which is what we got the other ways.

10. Sketch the pole diagram for $F(s) = (s+1)/(s^2 + 4s + 13)$. What does it tell you about the long-term behavior of the inverse transforms? Now invert the transform directly. Are these answers consistent?

The roots are $-2 \pm 3i$, so our function should behave roughly like $e^{-2t} \cos 3t$ in the long term: it decays like e^{-2t} while oscillating with angular pseudofrequency (pseudo-angular frequency?) 3.

Write

$$\frac{s+1}{s^2 + 4s + 13} = \frac{s+2}{(s+2)^2 + 9} - \frac{1}{3} \frac{3}{(s+2)^2 + 9}$$

The inverse transform of $s/(s^2+9)$ is $\cos 3t$, and of $3/((s+2)^2+9)$ is $\sin 3t$. So the solution is $e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$. This is consistent with the above.

11. Expand $3\cos(2t-\pi/3)$ as $a\cos(2t)+b\sin(2t)$. Write $\cos(3t)-\sin(3t)$ in standard form.

    It's $3\cos(2t)\cos(\pi/3)+3\sin(2t)\sin(\pi/3)=\frac{3}{2}\cos(2t)+\frac{3\sqrt{3}}{2}\sin(2t)$.

    For the second, we have $A=\sqrt{a^2+b^2}=\sqrt{2}$. Then $\cos\phi=a/A=1/\sqrt{2}$ and $\sin\phi=b/A=-1/\sqrt{2}$, so $\phi=-\pi/4$. The answer is then $\sqrt{2}\cos(3t+\pi/4)$.

12. How would you solve $\ddot{x}+7\dot{x}+12x=q(t)$ where $q(t)=t^2$, $e^t\cos t$, te^t, $(t-1)e^t$, or e^{-3t}?

    First: method of undetermined coefficients. Assume $x=at^2+bt+c$, plug in, solve for a,b,c.

    Second: complex replacement is $e^{(1+i)t}$. Then use ERF to find complex solution, and take the real part.

    Third: use variation of parameters to find a solution for $q(t)=te^t$. This I'll actually do. Guess $x(t)=u(t)e^t$. Then $\dot{x}(t)=\dot{u}(t)e^t+u(t)e^t$ and $\ddot{x}(t)=\ddot{u}(t)e^t+\dot{u}(t)e^t+\dot{u}(t)e^t+u(t)e^t$. So

    $$\begin{aligned}\ddot{x}+7\dot{x}+12x &= \left(\ddot{u}e^t+2\dot{u}e^t+ue^t\right)+\left(7\dot{u}e^t+7ue^t\right)+12ue^t\\ &= \left(\ddot{u}+9\dot{u}+20u\right)e^t.\end{aligned}$$

    We want this to be te^t, so $\ddot{u}+9\dot{u}+20u=t$. Guess $u(t)=at^2+bt+c$ and you find $u(t)=\frac{1}{20}t-\frac{9}{400}$. So

    $$x(t)=\left(\frac{1}{20}t-\frac{9}{400}\right)e^t.$$

    For $(t-1)e^t$ we can use the same approach, or save some trouble and recognize it as a time shift: $(t-1)e^t=(t-1)e^{t-1}\cdot e$. If $q(t)$ were just $(t-1)e^{t-1}$ then time shift gives

    $$x(t)=\left(\frac{1}{20}(t-1)-\frac{9}{400}\right)e^{t-1}.$$

    We need to account for the extra e factor, and so the solution we want is

    $$x(t)=\left(\frac{1}{20}(t-1)-\frac{9}{400}\right)e^{t-1}\cdot e=\left(\frac{1}{20}t-\frac{29}{400}\right)e^t$$

    At last: straight-up ERF for this one.

13. (a) A broken A/C unit has temperature $h(t) = \sin(\omega t)$, for some ω. Set up and solve a differential equation for the temperature of the room at time t. Is there any value of ω for which the temperature of the room is not bounded?

(Just leave everything in terms of ω and the coupling constant k, along with a constant c. I trust you could find c given initial conditions).

Newton's law of cooling gives $\dot{x}+kx = \sin(\omega t)$. We know a few ways to solve this. The homogeneous solution is $x_p = ce^{-kt}$. Complex replacement + ERF gives

$$x_p = Im\left(\frac{e^{i\omega t}}{k+i\omega}\right) = \frac{k\sin\omega t - \omega\cos\omega t}{k^2+\omega^2}.$$

Thus

$$x(t) = Ce^{-kt} + \frac{k\sin\omega t - \omega\cos\omega t}{k^2+\omega^2}.$$

(b) After a repair the air conditioner has temperature given by a standard square wave. Give a formula for $f(t)$.

The square wave has Fourier series given by

$$sq(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right).$$

By superposition our solution is then

$$\begin{aligned} f(t) &= ce^{-kt} + \frac{4}{\pi}\left(\frac{k\sin t - \cos t}{k^2+1^2} + \frac{k\sin 3t - 3\cos 3t}{3(k^2+3^2)} + \frac{k\sin 5t - 5\cos 5t}{5(k^2+5^2)} + \cdots\right) \\ &= ce^{-kt} + \frac{4}{\pi}\sum_{n \text{ odd}} \frac{k\sin nt - n\cos nt}{n(k^2+n^2)} \end{aligned}$$

(c) Suppose that the unit turns out at time $\pi/3$ instead of 0. Give a Fourier series for $f(t)$.

Time-shifting says we can just plug in $t-\pi/3$ for t.

$$\begin{aligned} f(t) &= ce^{-k(t-\pi/3)} + \frac{4}{\pi}\sum_{n \text{ odd}} \frac{k\sin n(t-\pi/3) - n\cos n(t-\pi/3)}{n(k^2+n^2)} \\ &= c_2e^{-kt} + \sum_{k \text{ odd}} \frac{k\left(\sin nt\cos\frac{n\pi}{3} - \cos nt\sin\frac{n\pi}{3}\right) - n\left(\cos nt\cos\frac{n\pi}{3} + \sin nt\sin\frac{n\pi}{3}\right)}{n(k^2+n^2)} \\ &= c_2e^{-kt} + \sum_{k \text{ odd}} \frac{k\cos\frac{n\pi}{3} - n\sin\frac{n\pi}{3}}{n(k^2+n^2)}\sin nt + \frac{-k\sin\frac{n\pi}{3} - n\cos\frac{n\pi}{3}}{n(k^2+n^2)}\cos nt. \end{aligned}$$

The coefficients are messy, but they're just numbers. So this is a legitimate Fourier series. Note that $\cos\frac{n\pi}{3}$ is periodic, so you could work out what this is for various n.

14. An operator has unit impulse response $w(t) = t^2u(t)$. What is $p(D)$? How can you solve $p(D)x = e^t$?

    We know $p(s) = 1/W(s)$, where $W(s)$ is the transfer function. So $2/s^3$, and it's $s^3/2$. You can get the solution via convolution:

$$x(t) = w(t) * q(t) = \int_{\tau=0}^{t} \tau^2 e^{t-\tau}\, d\tau = 2e^t - (t^2 + 2t + 2).$$

15. What is the Fourier series for $\sin^3(t)$?

    We need to expand this in terms of sine and cosine. There's a trig identity, which is not particularly memorable. To make do without, use complex replacement and some algebra tricks:

$$\begin{aligned}\sin^3(t) &= \left(\frac{e^{it} - e^{-it}}{2i}\right)^3 = \frac{1}{(2i)^3}\left(e^{3it} - 3e^{it} + 3e^{-it} - e^{-3it}\right)\\ &= \frac{1}{(2i)^2}\left(\frac{e^{3it} - e^{-3it}}{2i}\right) - \frac{3}{(2i)^2}\left(\frac{e^{it} - e^{-it}}{2i}\right)\\ &= \frac{3}{4}\sin t - \frac{1}{4}\sin(3t).\end{aligned}$$

    That's the Fourier series!

    In general, there are three ways to find the Fourier series of a function: i) compute the integrals for the coefficients (usually a pain) ii) relate it to functions for which we know the series (usually the square wave) iii) use trig identities (only applicable to things like the one above)