Math 18.06, r3/5
Problems #2
February 26, 2013

1. Consider the $2 \times 2$ matrix $A = \left[\begin{smallmatrix} 2 & -1 \\ 6 & -3 \end{smallmatrix}\right]$.
   (a) What is the column picture for $A\mathbf{x} = \mathbf{b}$?
   (b) For what vectors $\mathbf{b}$ is there a solution?
   (c) What are all solutions to $A\mathbf{x} = \mathbf{0}$?
   (d) Pick a non-zero $\mathbf{b}$ for which there is a solution. What are all solutions to $A\mathbf{x} = \mathbf{b}$? Sketch the set of solutions.
   (e) Every solution in (c) is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_p$ is some fixed particular solution. How does $\mathbf{x}$ vary when we pick different $\mathbf{x}_n$ in the nullspace?

2. Let $A$ be the matrix
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 & 8 \\ 0 & 0 & 6 & 6 & 6 \end{bmatrix}.$$
   (a) Put $A$ in echelon form, and reduced echelon form.
   (b) What are the free and pivot variables? Find a special solution for each free variable.
   (c) Find all solutions to $A\mathbf{x} = \mathbf{0}$.
   (d) What is the rank of $A$? What does this tell you about the rows and columns? Which columns are combinations of earlier columns?
   (e) What are the possible ranks of a $3 \times 5$ matrix? What are the possible numbers of special solutions of a $3 \times 5$ matrix?
   (f) For what vectors $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?
   (g) Find all solutions to $A\mathbf{x} = \left(\begin{smallmatrix} 1 \\ 4 \\ 6 \end{smallmatrix}\right)$.

3. Which of the following subsets of $\mathbb{R}^3$ are subspaces?
   (a) The line parameterized by $\mathbf{v}(t) = (1, 2, 3) + (1, 0, 0)\,t$.
   (b) The line parameterized by $\mathbf{v}(t) = (1, 2, 3) + (2, 4, 6)\,t$.
   (c) The set of points at distance 1 from the origin.
   (d) The plane defined by the equation $x + 2y + 3z = c$ (for different values of $c$).

   For those which are not subspaces, what is the span of the vectors in the set?

4. (a) Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, and $\mathbf{v}_3 = (7, 8, 9)$. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, but $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.
   (b) What is a basis for the space of symmetric $3 \times 3$ matrices? What is the dimension of this space?
   (c) How about for upper triangular $3 \times 3$ matrices?
   (d) Give a basis for the set of $3 \times 3$ upper-triangular matrices with trace 0 (i.e. such that the sum of the diagonal elements is 0).

5. ($*$) This is a challenge problem to highlight the analogy with the principle of superposition in 18.03 – it doesn't have too much to do with what we're studying now, so don't worry about it!

   Let $V$ be the set of all infinitely differentiable functions of period $2\pi$.

   (a) Check that $V$ is a vector space.

   (b) Check that the map $D = \frac{\partial}{\partial t^2} + I$ gives a linear map from $V$ to $V$.

   (c) What is the nullspace of $D$?

   (d) What are all solutions in $V$ of $Df = \cos 2t$? $Df = \cos t$?

   (e) Represent the function $f$ as a vector, with entries its Fourier coefficients. What is the $\infty \times \infty$ "matrix" for $D$? Can you describe the range of $D$?