Math 18.06, r3/5
Problems #1
February 12, 2013

1. *Consider the system of two linear equations $x - y = 0$, $x + y = 2$.*

   (a) *Express this sytem in matrix form.*
   We want to solve
   $$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

   (b) *What is the "row picture" for this system?*
   Each row of the system defines a line. The solution is the intersection of the two lines $x - y = 0$, $x + y = 2$.

   (c) *What is the "column picture"?*
   We want to write $\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$ as a linear combination of the two vectors $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$.

2. *Consider the matrix* $A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 5 & 1 \\ 2 & -1 & 6 \end{bmatrix}$.

   (a) *What is the linear system of equations corresponding to* $A\mathbf{x} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$*?*
   This is three linear equations in three variables: $2x+3y = 3$, $4x+5y+z = 7$, $2x-y+6z = 5$.

   (b) *Solve the system using elimination.*

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 3 \\ 4 & 5 & 1 & 7 \\ 2 & -1 & 6 & 5 \end{array}\right] \xrightarrow{\mathbf{r}_2 -= 2\mathbf{r}_1} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 3 \\ 0 & -1 & 1 & 1 \\ 2 & -1 & 6 & 5 \end{array}\right] \xrightarrow{\mathbf{r}_3 -= \mathbf{r}_1}$$
$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & -4 & 6 & 2 \end{array}\right] \xrightarrow{\mathbf{r}_3 -= 4\mathbf{r}_2} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

   Now we back substitute: $z = -1$, $y = -2$, $x = 9/2$.

   (c) *Give an LU decomposition for A.*
   L will be the product of the matrices for our row maneuvers, and U the matrix we're left with at the end. Remember that $E_{32}E_{31}E_{21}A = U$, so $A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U = LU$. So

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

   The decomposition is now $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

3. (a) *Write down a matrix A which projects 3-dimensional vectors onto the 2-dimensional xy-plane. What are the dimensions of A?*

   Use $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$; then $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. This maps from a 3-dimensional space to a 2-dimensional one, and its dimensions are 2×3 as expected.

   (b) *Write down a 2×2 matrix B which rotates vectors $90°$ counterclockwise, and a 3×3 matrix which rotates $90°$ counterclockwise around the z-axis without changing the height.*

   This is $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For the second, it's $B' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (c) *What is a 2×3 matrix C that projects vectors to the xy-plane and then rotates them $90°$ clockwise?*

   Take $C = BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The product of two matrices first does the transformation of the one on the right, followed by the transformation of the one on the left.

   (d) *Let A be the matrix from part (a). For what 3×3 matrices D is it true that $A\mathbf{v} = A(D\mathbf{v})$ for all vectors $\mathbf{v}$?*

   D can't change the x or y part of a vector, but it can send z to any combination of x, y, and z. Thus D must be of the form

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}.$$

4. *Consider the matrix*

$$M = \begin{bmatrix} 1 & c & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Suppose you try to solve $M\mathbf{x} = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ using elimination. For what value(s) of c does this fail? Interpret this in terms of the row and column pictures for the linear system.

$$\left[\begin{array}{ccc|c} 1 & 0 & c & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{\mathbf{r}_2 -= \mathbf{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & c & 1 \\ 0 & 1 & -c & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{\mathbf{r}_3 -= \mathbf{r}_2} \left[\begin{array}{ccc|c} 1 & 0 & c & 1 \\ 0 & 1 & -c & 1 \\ 0 & 0 & 1+c & 2 \end{array}\right].$$

Everything is OK, unless $c = -1$! In that case there is no third pivot and we can't solve. In terms of the row picture: as we change c, the hyperplanes move around in space. When c hits -1, the three planes no longer intersect; the three lines consisting of the intersection of two of the planes are parallel (I think M^c^Kernan called this the "Toblerone configuration" in 18.02?). We have $\mathbf{r}_1 - \mathbf{r}_2 + \mathbf{r}_3 = \mathbf{0}$.

In terms of the column picture: we have three vectors in space. They are linearly independent, except when $c = -1$, when they all lie in a plane (and so too do all combinations of them). We have $\mathbf{c}_1 + \mathbf{c}_2 - \mathbf{c}_3 = \mathbf{0}$.

5. *Use Gauss-Jordan elimination to compute A^{-1}, where A is the matrix from the second problem. Use this to solve the system in problem 2(b) again.*

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 & 1 & 0 \\ 2 & -1 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\mathbf{r}_2 -= 2\mathbf{r}_1} \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 2 & -1 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\mathbf{r}_3 -= \mathbf{r}_1}$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & -4 & 6 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\mathbf{r}_3 -= 4\mathbf{r}_2} \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 7 & -4 & 1 \end{array}\right] \xrightarrow{\mathbf{r}_2 -= 1/2\,\mathbf{r}_3}$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -\frac{11}{2} & 3 & -\frac{1}{2} \\ 0 & 0 & 2 & 7 & -4 & 1 \end{array}\right] \xrightarrow{\mathbf{r}_1 += 3\mathbf{r}_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -\frac{31}{2} & 9 & -\frac{3}{2} \\ 0 & -1 & 0 & -\frac{11}{2} & 3 & -\frac{1}{2} \\ 0 & 0 & 2 & 7 & -4 & 1 \end{array}\right] \xrightarrow[\text{pivots}]{\text{divide}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{31}{4} & \frac{9}{2} & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{11}{2} & -3 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{7}{2} & -2 & \frac{1}{2} \end{array}\right].$$

The inverse is the thing on the right:

$$A^{-1} = \begin{bmatrix} -\frac{31}{4} & \frac{9}{2} & -\frac{3}{4} \\ \frac{11}{2} & -3 & \frac{1}{2} \\ \frac{7}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

To re-solve the equation, work out $A^{-1}\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$.

6. *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are two vectors. Let* A *be the matrix whose columns are* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{u}+\mathbf{v}$. *Let* B *be the matrix whose rows are* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{u}+\mathbf{v}$. *Can you find a vector* $\mathbf{x}$ *such that* $A\mathbf{x} = \mathbf{0}$? *Such that* $B\mathbf{x} = \mathbf{0}$?

   To get $A\mathbf{x} = 0$, we need $\mathbf{x}$ to encode a linear combination of the columns which is $\mathbf{0}$. But $(1)(\mathbf{u}) + (1)(\mathbf{v}) + (-1)(\mathbf{u}+\mathbf{v}) = 0$, so we can use the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

   To get $B\mathbf{x} = 0$, we need $\mathbf{x}$ to satisfy $\mathbf{x}\cdot\mathbf{u} = 0$, $\mathbf{u}\cdot\mathbf{v} = 0$, and $\mathbf{x}\cdot(\mathbf{u}+\mathbf{v}) = 0$. The first two of these imply the third, so we're in good shape as long as our $\mathbf{x}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. One vector which does the trick is $\mathbf{x} = \mathbf{u}\times\mathbf{v}$.

7. *Can you find any* 2×2 *matrices with the following properties? Can you find all of them? (Hint: there may be none)*

   (a) $A\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$ *for any* $\mathbf{u}$ *and* $\mathbf{v}$.

   The only such A is the identity matrix. Taking $\mathbf{v} = (1,0)$, we see that A doesn't change the first coordinate of any vector $\mathbf{u}$. Taking $\mathbf{v} = (0,1)$, we see that it doesnt' change the second. The only such matrix is I.

   (b) $A\mathbf{u}\cdot A\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$ *for any* $\mathbf{u}$ *and* $\mathbf{v}$.

   If A is any rotation matrix, it will have this property. So will a reflection matrix, or any product of the two; we'll study matrices with this property later in the course - they are the *orthogonal matrices.*

   (c) A *is a* 2×2 *matrix such that* $A\mathbf{u}$ *is a vector of length* $|\mathbf{u}|$ *in the direction* $(1,0)$.

   This is impossible; the transformation isn't linear. Such a matrix must send $(1,0)$ to $(1,0)$ and $(0,1)$ to $(1,0)$. By linearity it would then have to send $(1,1)$ to $(2,0)$, which isn't what we want.

8. *Roughly how many addition and multiplication operations are required to compute the inverse of an $n \times n$ matrix using Gauss-Jordan elimination?*

   First we do Gaussian elimination to get the matrix in upper triangular form. Clearing the first pivot takes n rows, with $2n$ multiplications and $2n$ subtractions on each. We'll count this as n^2 of each operation. To clear the second, it's about $n-1$ rows and $(n-1)+n$ of each. Total, we have $\sum_{k=0}^{n}(n-k)(2n-k)$. This is on the order of n^3 operations (page 99 in the book is a bit more detailed). Next, we clear above the diagonal, which is even easier. At last, we divide through by the pivots. This is about n^2 more multiplications, which is negligible in the grand scheme of things. All told it will take some multiple of n^3.

   Note: in 18.02 we saw another method for finding the inverse of a 3×3 matrix, by computing the determinant and then shuffling around the entries and flipping some signs. This does generalize to $n \times n$ matrices, but computing the determinant of a large matrix 18.02-style is a slow process, and Gauss-Jordan will be faster. It's not the best, though: Gauss-Jordan takes $O(n^3)$ operations, while the slickest algorithms do a little better than $O(n^{2.4})$.