Math 18.06, r3/5
Problems #2
February 26, 2013

1. *Consider the 2×2 matrix $A = \left[\begin{smallmatrix} 2 & -1 \\ 6 & -3 \end{smallmatrix}\right]$.*

   (a) *What is the column picture for $A\mathbf{x} = \mathbf{b}$?*

   We want to write $\mathbf{b}$ as a combination of $(2,6)$ and $(-1,-3)$. For most choices of $\mathbf{b}$, this isn't going to happen.

   (b) *For what vectors $\mathbf{b}$ is there a solution?*

   From the above equations, we can see they intersect if $b_2 = 3b_1$. In terms of the matrix, this is the same thing as $\mathbf{b}$ being in the column space of A, which is one-dimensional.

   (c) *What are all solutions to $A\mathbf{x} = \mathbf{0}$?*

   The nullspace is spanned by $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. The solutions are all multiples $c\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$.

   (d) *Pick a non-zero $\mathbf{b}$ for which there is a solution. What are all solutions to $A\mathbf{x} = \mathbf{b}$? Sketch the set of solutions.*

   Let's use $\mathbf{b} = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$. One particular solution is given by $\mathbf{x} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. The general solution is

   $$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$$

   (e) *Every solution in (c) is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_p$ is some fixed particular solution. How does $\mathbf{x}$ vary when we pick different $\mathbf{x}_n$ in the nullspace?*

   As c changes, the solution $\mathbf{x}$ moves along a line.

2. *Let A be the matrix*

   $$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 & 8 \\ 0 & 0 & 6 & 6 & 6 \end{bmatrix}.$$

   (a) *Put A in echelon form, and reduced echelon form.*

   $$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 & 8 \\ 0 & 0 & 6 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   For rref, keep going: divide rows by numbers to make the pivots 1, and put 0s above the pivots.

   $$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (b) *What are the free and pivot variables? Find a special solution for each free variable.*

   The variables x_1 and x_3 are pivot variables, since there are nonzero pivots in these columns. x_2, x_4, x_5 are free variables, and we'll have three special solutions in $N(U)$.

   - First take $x_2 = 1$, $x_4 = 0$, $x_5 = 0$. We get $x_1 + 2 = 0$ and $x_3 = 0$, so the solution is $(-2, 1, 0, 0, 0)$.

- Now take $x_2 = 0$, $x_4 = 1$, $x_5 = 0$. Here $x_1 + 1 = 0$ and $x_3 + 1 = 0$, and the solution is $(-1, 0, -1, 1, 0)$.
- At last, try $x_2 = 0$, $x_4 = 0$, and $x_5 = 1$. Then $x_1 + 2 = 0$ and $x_3 + 1 = 0$, with solution $(-2, 0, -1, 0, 1)$

(c) *Find all solutions to* $A\mathbf{x} = \mathbf{0}$.

The nullspace contains all linear combinations of the special solutions. Note that $N(A) = N(U) = N(R)$. In this case that means

$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(d) *What is the rank of* A*? What does this tell you about the rows and columns? Which columns are combinations of earlier columns?*

The rank of A is the number of pivots. In this case, it's 2. There are only two independent columns, and two independent rows. When we put the matrix in echelon form, there will be a row of zeroes.

Remember an important fact: the free columns are combinations of earlier pivot columns, and the coefficients used in writing these combinations are the same for A as for the echelon forms U and R. We can read off these coefficients from the special solutions we already found: $\mathbf{c}_2 = 2\mathbf{c}_1$, $\mathbf{c}_4 = \mathbf{c}_1 + \mathbf{c}_3$, $\mathbf{c}_5 = 2\mathbf{c}_1 + \mathbf{c}_3$.

(e) *What are the possible ranks of a* 3×5 *matrix? What are the possible numbers of special solutions of a* 3×5 *matrix?*

The maximum number of pivots of a matrix with these dimensions is 3, and the rank can be 0, 1, 2, or 3.

Every 3×5 matrix has at least $5 - 3 = 2$ special solutions, because there can be at most three pivot variables. It's also possible to have more solutions: this example has 3, and other matrices can have 4 or 5 (for the zero matrix).

(f) *For what vectors* $\mathbf{b}$ *does* $A\mathbf{x} = \mathbf{b}$ *have a solution?*

There's a solution as long as $\mathbf{b}$ is in the column space of A. The column space is the span of the two pivot columns. Important: although A and R have the same row space, they don't have the same column space!

(g) *Find all solutions to* $A\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$.

To find the complete solution, we need to find one particular solution and then we can add any vector that's in the nullspace.

There are two basic ways to find a particular solution: a) look at the matrix and guess one b) use an augmented matrix and elimination. Maybe you can guess a solution here, but let's eliminate anyway:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 6 & 7 & 8 & 4 \\ 0 & 0 & 6 & 6 & 6 & 6 \end{array}\right].$$

Carry out the same elimination steps as above, and you get the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 6 & 7 & 8 & 4 \\ 0 & 0 & 6 & 6 & 6 & 6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & 6 & 6 & 6 & 6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to$$
$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last row is $\mathbf{0} = \mathbf{0}$, as expected. At this point, the easiest approach is to plug in 0 for each of the free variables and solve for the pivot variables. The solution we get is $(-2, 0, 1, 0, 0)$. The general solution is now this particular solution plus anything in the nullspace:

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

3. *Which of the following subsets of $\mathbb{R}^3$ are subspaces?*

   (a) *The line parameterized by $\mathbf{v}(t) = (1, 2, 3) + (1, 0, 0)\,t$.*

   (b) *The line parameterized by $\mathbf{v}(t) = (1, 2, 3) + (2, 4, 6)\,t$.*

   Let's do these two together. A line in $\mathbb{R}^3$ is a subspace if and only if it goes through the point $(0, 0, 0)$. The first of these does not, while the second does. The point $(1, 2, 3)$ is on the first line, but $2 \cdot (1, 2, 3) = (2, 4, 6)$ is not.

   (c) *The set of points at distance 1 from the origin.*

   This one isn't a subspace either. $(1, 0, 0)$ is in this subset, but $2 \cdot (1, 0, 0) = (2, 0, 0)$ isn't.

   (d) *The plane defined by the equation $x + 2y + 3z = c$ (for different values of c).*

   A plane is a subspace if and only if it goes through the point $(0, 0, 0)$. This will be the case if and only if $c = 0$.

For those which are not subspaces, what is the span of the vectors in the set?

For (a), there is a plane through $(0, 0, 0)$ which contains this entire line. It is the unique plane through the three points $(0, 0, 0)$, $(1, 2, 3)$, and $(2, 2, 3)$. For (b), the line was already a subspace. For (c), this subset contains the three vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, so the span is the whole space. For (d), the span is the plane itself if $c = 0$, or all of $\mathbb{R}^3$ if $c \neq 0$.

4. (a) *Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, and $\mathbf{v}_3 = (7, 8, 9)$. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, but $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.*

   Say $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$; we need to show that $a = b = 0$. A solution must have $1 \cdot a + 4 \cdot b = 0$, $2 \cdot a + 5 \cdot b = 0$, and $3 \cdot a + 6 \cdot b = 0$. The first two equations alone have the single solution $a = b = 0$.

   On the other hand, $(1)\mathbf{v}_1 + (-2)\mathbf{v}_2 + (1)\mathbf{v}_3$ is a nontrivial linear relation among the three vectors, so the three are not linearly independent.

   (b) *What is a basis for the space of symmetric 3×3 matrices? What is the dimension of this space?*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

   The dimension of the space is equal to the number of vectors in a basis, which is 6.

(c) *How about for upper triangular 3×3 matrices?*

There is again a six-dimensional space:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}.$$

(d) *Give a basis for the set of 3×3 upper-triangular matrices with trace 0 (i.e. such that the sum of the diagonal elements is 0).*

$$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}.$$

To see that this is a basis, we need to know two things: that the vectors are linearly independent, and that every 3×3 upper triangular matrix with trace 0 is a linear combination of these six. Such a matrix looks like

$$A = \begin{bmatrix}a&d&e\\0&b&f\\0&0&c\end{bmatrix},$$

with $a+b+c=0$. We can write it as

$$\begin{bmatrix}a&d&e\\0&b&f\\0&0&c\end{bmatrix} = d\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} + e\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} + d\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$$
$$+ (-b)\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix} + (-c)\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}.$$

This combination clearly has all the right entries except in the top left, where it has $-b-c$ and we want a. But since $a = -b-c$ by assumption, everything is A-ok.

5. *($*$) This is a challenge problem to highlight the analogy with the principle of superposition in 18.03 – it doesn't have too much to do with what we're studying now, so don't worry about it!*

Let V be the set of all infinitely differentiable functions of period 2π.

(a) *Check that V is a vector space.*

Add two such functions or multiply one by a scalar, and you get another such function. No problems here.

(b) *Check that the map $D = \frac{\partial}{\partial t^2} + I$ gives a linear map from V to V.*

This is just basic properties of the derivative. If a and b are scalars:

$$D(af+bg) = \frac{\partial^2(af+bg)}{\partial t^2} + (af+bg) = a\left(\frac{\partial^2 f}{\partial t^2} + f\right) + b\left(\frac{\partial^2 g}{\partial t^2} + g\right) = a\,D(f) + b\,D(g)$$

(c) *What is the nullspace of D?*

These are the functions that solve $\frac{\partial^2 f}{\partial t^2} + f = 0$, namely the linear combinations of $\cos t$ and $\sin t$, i.e. $f(t) = c_1 \cos t + c_2 \sin t$.

(d) *What are all solutions in V of $Df = \cos 2t$? $Df = \cos t$?*

Thinking back to 18.03, the general solution is a solution of the homogeneous equation, plus a particular solution. For $\cos 2t$, a particular solution is $-\frac{1}{3}\cos 2t$, and the general solution is $-\frac{1}{3}\cos 2t + c_1 \cos t + c_2 \sin t$. For $\cos t$, we have resonance, and there isn't a periodic solution. So there is no solution in V.

(e) *Represent the function f as a vector, with entries its Fourier coefficients. What is the $\infty \times \infty$ "matrix" for D? Can you describe the range of D?*

Write $f = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$. We have $D(\cos(nt)) = (1-n^2)\cos(nt)$ and $D(\sin(nt)) = (1-n^2)\sin(nt)$. The matrix, with respect to the basis $1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots$, is

$$D = \begin{pmatrix} 1 & & & & & & & & \\ & 0 & 0 & & & & & & \\ & 0 & 0 & & & & 0 & & \\ & & & -3 & 0 & & & & \\ & & & 0 & -3 & & & & \\ & & & & & \ddots & & & \\ & & 0 & & & & 1-n^2 & 0 & \\ & & & & & & 0 & 1-n^2 & \\ & & & & & & & & \ddots \end{pmatrix}.$$

The span (those g for which there is a solution to $Df = g$) is given by all g with 0 coefficient for $\cos t$ and $\sin t$.