Math 18.06, r3/5
Problems #6
April 3, 2013

1. (a) *Say you try to use Cramer's rule to solve* $A\mathbf{x} = \mathbf{b}$*, where* $\mathbf{b}$ *is equal to one of the columns of* A*. What happens?*

   We know that the solution is $x_i = \frac{\det B_i}{\det A}$. If i is anything except the number of the column equal to $\mathbf{b}$, then B_i has two columns equal and the determinant is 0. For the column where they do coincide, B_i is equal to A, and we get $x_i = 1$. This is obviously the right solution.

   (b) *Suppose that* A *is a matrix with integer entries, and* $\mathbf{b}$ *is a vector with integer entries. Under what conditions on* A *must the solutions to* $A\mathbf{x} = \mathbf{b}$ *also have integer entries? Explain this using Cramer's rule.*

   This will work if $\det A$ is ± 1, since then $\det B_i / \det A$ are all integers.

   (c) *Compute the area of a triangle with vertices* $(1,0,0)$*,* $(0,1,0)$*, and* $(0,0,1)$*.*

   Shift everything up down by $(-1,0,0)$, and it's the same area as the triangle with vertices $(0,0,0)$, $(-1,1,0)$, and $(-1,0,1)$. This is given by $A = \frac{1}{2}\left|(-1,1,0)\times(-1,0,1)\right| = \left|\mathbf{i}+\mathbf{j}+\mathbf{k}\right|/2 = \sqrt{3}/2$.

   (d) *Consider the matrix*

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

   Compute $\det A$ *by each of the three main methods: cofactor expansion, the "big formula", and multiplication of pivots. What are the eigenvalues and eigenvectors?*

   This matrix isn't too hard no matter which method we use.

   Elimination is done after subtracting half the first row from the second, and the pivots are 2, 3/2, 2, and 4. The determinant is the product, which is 24.

   For cofactor expansion, it's easiest to use the last row (or the third column). This gives $\det A = 4 \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 4(8-2) = 24$.

   For the big formula, there are only two terms that aren't 0: $2\cdot 2\cdot 2\cdot 4 - 1\cdot 1\cdot 2\cdot 4 = 24$.

   To find eigenvalues:

$$\begin{aligned}\det(A-\lambda I) &= \det \begin{pmatrix} 2-\lambda & 1 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 4-\lambda \end{pmatrix} \\ &= (2-\lambda)(4-\lambda)\left((2-\lambda)^2 - 1\right) \\ &= (2-\lambda)(4-\lambda)(\lambda^2 - 4\lambda + 3) = (2-\lambda)(4-\lambda)(1-\lambda)(3-\lambda).\end{aligned}$$

The eigenvalues are therefore 1, 2, 3, 4. To find the eigenvector for $\lambda = 1$, we look at

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

An eigenvector will be something in the nullspace of this matrix, and we know how to find vectors in the nullspace via special solutions. In this instance our special solution is $\mathbf{v} = (-1, 1, 0, 0)$. The other eigenvectors are computed in the same way.

2. *Suppose that A is an upper-triangular matrix. What does the cofactor matrix C look like? What does this tell you about A^{-1}?*

   Think about what happens when you take the C_{ij} cofactor, where ij is an entry in the upper triangular half. When you cross out row i and column j, what's left is definitely upper-triangular. Not only that, but the old $i + 1$st row is the new ith one. This row starts with i 0's, so what's left actually has a zero on the diagonal. Since it's upper triangular, the determinant is 0. So every entry of C above the main diagonal is 0, i.e. C lower-triangular.

   We know that $A^{-1} = C^T / \det A$. Since C^T is upper-triangular, so too is A^{-1}.

3. *Let A_n be the $n \times n$ matrix shown below:*

$$A_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

   Let $b_n = \det A_n$. Express b_n in terms of b_{n-1}. Can you find a formula for b_n?

   Expand by cofactors down the second column (you could use other columns too, e.g. the last column. You'll get the same answer no matter what).

$$b_n = (-1) \cdot 1 \cdot \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

   The term on the left has only a single nonzero summand in the big formula for the determinant; in fact it's lower-triangular. The determinant is 2^{n-2} (what I've written out is the case $n = 5$, and there are three 2's along the diagonal). The matrix on the right is just A_{n-1}.

   So our formula is

$$b_n = 2b_{n-1} - 2^{n-2}$$

We start with $b_3 = 0$. The next few terms are $b_4 = -2^2 = -4$, $b_5 = 2(-4) - 2^3 = -8 - 8 = -16$, $b_6 = 2(-16) - 32 = -48$. Finding the general formula isn't really an 18.06 problem, though it turns out to be $b_n = -(n-3)2^{n-2}$, contrary to what I claimed in recitation.

4. *Suppose that A is a 2×2 matrix with eigenvalues λ_1 and λ_2. What are the trace and determinant of A? What are the trace and determinant of A^2? Can you express these in terms of* $\det A$ *and* $\operatorname{tr} A$*?*

   The determinant is $\det A = \lambda_1\lambda_2$, which I'll call d. The trace is $\operatorname{tr} A = \lambda_1 + \lambda_2$, hereafter called t.

   We know $\det(A^2) = \det A \cdot \det A = d^2$. For the trace, the key observation is that the eigenvalues of A^2 are λ_1^2 and $= \lambda_2^2$, and so

$$\operatorname{tr} A^2 = \lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = t^2 - 2d.$$

5. *Suppose that $B(t)$ is the following matrix (which depends on a parameter t):*

$$\begin{pmatrix} t & t^2 & 1 & 2 \\ 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

   For how many values of t do you expect that $\det B(t) = 0$*? What is* $\frac{d}{dt}\det(B(t))$*?*

   The cofactor expansion across the top row is

$$\det B(t) = tC_{11} + t^2C_{12} + C_{13} + C_2.$$

   The precise values of the cofactors aren't so interesting to us. What matters is that this is a quadratic in t, so more likely than not it has two real solutions (to actually check this you'd need to be sure it doesn't have a double root). The derivative is $2tC_{12} + C_{11}$, a linear function.

6. *Given three vectors* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{w}$ *in* $\mathbb{R}^4$*, define the "cross product" of the three vectors to be*

$$\mathbf{u} \boxtimes \mathbf{v} \boxtimes \mathbf{w} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}.$$

   Explain why $\mathbf{u} \boxtimes \mathbf{v} \boxtimes \mathbf{w}$ *is perpendicular to each of the three vectors.*

   For example,

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \boxtimes \mathbf{v} \boxtimes \mathbf{w}) &= (u_1, u_2, u_3, u_4) \cdot (C_{11}, C_{12}, C_{13}, C_{14}) \\ &= u_1C_{11} + u_2C_{12} + u_3C_{13} + u_4C_{14} \\ &= \det \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} = 0. \end{aligned}$$

7. *Give an example of a 2×2 matrix with both eigenvalues equal, which isn't a multiple of the identity. Geometrically, what does your matrix do to a vector? What are the eigenvectors?*

   One example is $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. This rescales everything by a factor of λ, then shifts y by the original x; it's a kind of "shear" transformation. In this example, there's only a single eigenvector, $(1, 0)$.

8. *Use the big formula to explain why* $\det A = \det A^T$ *for a square matrix* A.

   We know that

$$\det A = \sum_P \det(P) a_{1\alpha_1} \cdots a_{n\alpha_n}.$$

   The ways to choose one thing from each row and each column of A are exactly the same as the ways to do this for A^T. The plus and minus signs coincide too, since the determinant of P is equal to the determinant of P^T. The sum appearing in the big formula is exactly the same either way.