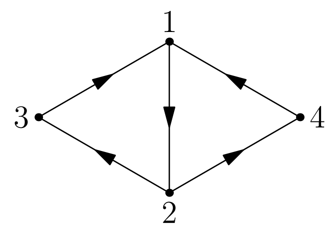
Math 18.06, r3/5
Problems #7
April 9, 2013

1. *Let G be the following graph with six nodes.*



Write down the incidence matrix A for G. What is the nullspace of A? What is the left nullspace of A, and what does this tell you about the graph?

We have

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

The nullspace of the matrix consists of $(1, 1, 1, 1, 1, 1)$ only, as is always the case with these things.

To find the left nullspace, you need to do elimination on A^T. In this case you'd find

$$\text{rref}(A^T) = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can see that $r_1 + r_2 + r_3 = 0$ and $r_1 + r_4 + r_5 = 0$. The two vectors $(1, 1, 1, 0, 0)$ and $(1, 0, 0, 1, 1)$ are a basis for the left nullspace. This corresponds to the fact that edges $1, 2, 3$ and $1, 4, 5$ above form loops in the graph.

2. *Find an orthonormal basis for the plane in $\mathbb{R}^3$ defined by $x + y + z = 0$. Use your basis to find the matrix for projection onto this plane. Identify the nullspace and column space of P.*

First we'd better find *a* basis. This we know how to do: this plane is the nullspace of the matrix $(1\ 1\ 1)$, and a basis is given by the two special solutions $(-1, 1, 0)$ and $(-1, 0, 1)$. This isn't an orthogonal basis, however. For that, we need Gram-Schmidt.

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2^T \mathbf{w}_1}{\mathbf{w}_1^T \mathbf{w}_1}\mathbf{w}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

The projection is $P = QQ^T$ (remember that our matrix is orthogonal), which is

$$P = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

3. *Use the cofactor matrix to find the inverse of*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

We'll use the formula $A^{-1} = \frac{C^T}{\det A}$. The cofactor matrix is

$$C = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$$

Since $\det A = 1$, this gives

$$A^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

4. *Find the best fit line through the three points* $(0, 0)$, $(1, 0)$, $(1, 1)$.

The corresponding system is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The best fit is $A^T A\mathbf{x} = A^T\mathbf{b}$. $A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$ This has inverse $\begin{pmatrix} 3/2 & -1 \\ -1 & 1 \end{pmatrix}$, so $\hat{\mathbf{x}} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$.
This is the line $y = \frac{1}{2}x + 0$.

5. (a) *Consider a parallelogram with vertices at* $(0,0)$, $(a,0)$, (c,d), *and* $(a+c,d)$. *What is its area?*

   It's
   $$\det\begin{pmatrix} a & c \\ 0 & d \end{pmatrix} = ad,$$
   which agrees with the formula base $\times$ height.

   (b) *Let* A *be a* 3×3 *matrix with determinant* 5. *What is the determinant of* $-2A^T$*?*

   Multiplying every row by -2 multiplies the determinant by $(-2)^3 = -8$. Taking transpose doesn't change anything.

   (c) *Suppose that* A *is symmetric and* P *is the projection onto the nullspace of* A. *What is* AP*?* PA*?*

   We have $PA = P^T A^T = (AP)^T$. But but $A(P\mathbf{v}) = 0$ for any $\mathbf{v}$, since $P\mathbf{v}$ is in the nullspace of A. So $AP = 0$.

6. *Consider the following very small-dimensional instance of least squares fit. Let* A *be the* 2×1 *matrix* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, *and* $\mathbf{b}$ *be the vector* $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

   Find the least-squares solution $A\hat{\mathbf{x}} = \mathbf{p}$ *to* $A\mathbf{x} = \mathbf{b}$. *Compute the vectors* $\mathbf{p}$ *and* $\mathbf{e} = \mathbf{b} - \mathbf{p}$. *What's the relation between* $\mathbf{e}$ *and* A*?*

   Replace the system with $A^T A\mathbf{x} = A^T\mathbf{b}$. This is the 1×1 system $(5)\mathbf{x} = (4)$, solved by $\hat{\mathbf{x}} = 4/5$. Then $\mathbf{p} = A\hat{\mathbf{x}} = (4/5, 8/5)$. The error is $\mathbf{e} = (-4/5, -2/5)$. This is perpendicular to the column space of A (which is to say that $A^T\mathbf{e} = 0$).

7. *Let*
   $$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$
   What is $\det A$*? What is the* $(1,2)$*-entry of* A^{-1}*?*

   Cofactors across the first row gives $\det A = 1(2) - 2(-3) = 8$. Since $A^{-1} = C^T/\det A$, the $(1,2)$ entry of A^{-1} is given by the $(2,1)$ entry of C, divided by 8. This entry is is $(-1)\det\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = (-1)(-2) = 2$. So the entry in question is $1/4$.

8. *Suppose that* A *is an* $m\times n$ *matrix whose rows are linearly independent. Compute the dimension of the nullspace of* $A^T A$.

   For a matrix with linearly independent columns, $M^T M$ is invertible. In our case, A^T has independent columns, so $(A^T)^T A^T = AA^T$ is invertible. This is an invertible $m\times m$ matrix and so has rank m. Now, the rank of a matrix is equal to the rank of its transpose. The transpose of AA^T is $A^T A$, so this too has rank m. This is an $n\times n$ matrix, and the dimension of the nullspace is therefore $n - m$.