Math 18.06, r3/5
Problems #8
April 23, 2013

1. (a) *An example of a matrix that isn't diagonalizable is:* ________________________.
   One is
   $$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
   Both eigenvalues are 1. If you try to find the eigenvectors corresponding to this eigenvalue, you'll quickly discover that $(1, 0)$ is the only one, and we can't find a set of independent eigenvectors.

   (b) *If A is symmetric, then it can be diagonalized as* ________________________. *Is every matrix that can be diagonalized in this way symmetric?*
   It diagonalizes as $A = Q\Lambda Q^T$. Anything of this form is symmetric: $(Q\Lambda Q^T)^T = Q\Lambda Q^T$.

   (c) *We had another decomposition for symmetric matrices, $A = LDL^T$, with D diagonal. Is this the same thing as the diagonalization of A?*
   No! There D was a matrix with the pivots on the diagonal. The pivots and eigenvalues aren't the same thing. D and Λ will be diagonal matrices with the same determinant and number of negative entries, though.

   (d) *When is a diagonal matrix positive-definite? Explain why $\mathbf{x}^T D\mathbf{x}$ could be negative if D has a negative entry.*
   This is when all the entries are positive. If the top left entry is negative, we'd get a negative answer when plugging in $\mathbf{x} = (1, 0, 0, 0, 0, 0)$.

2. *Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Is A diagonalizable? If so, diagonalize it using an orthogonal eigenbasis.*

   The eigenvalues are 1, 1, and 3. We need to find three independent eigenvectors. For 3, it's $(0, 1/\sqrt{2}, 1/\sqrt{2})$. For 1, we need to pick two independent eigenvalues. These should be two things in the nullspace of
   $$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$
   Any (x, y, z) with $y + z = 0$ will be OK. We can use $(1, 0, 0)$ and $(0, 1/\sqrt{2}, -1/\sqrt{2})$, for example. Then
   $$A = Q\Lambda Q^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

3. *Prove that if A is a skew-symmetric real matrix ($A^T = -A$), then all the eigenvalues of A are purely imaginary. Here's a strategy:*

   (a) *Prove that $A\mathbf{x} \cdot \mathbf{y} = -\mathbf{x} \cdot A\mathbf{y}$ for any vectors $\mathbf{x}$ and $\mathbf{y}$, where $\cdot$ is the complex dot product $\mathbf{x} \cdot \mathbf{y} = \bar{\mathbf{x}}^T \mathbf{y}$.*

   (b) *Suppose that $\mathbf{x}$ is an eigenvector with eigenvalue λ. Plug this in for $\mathbf{y}$ in the above and conclude that $\bar{\lambda} = -\lambda$.*

   We have

$$A\mathbf{x} \cdot \mathbf{y} = (\overline{A\mathbf{x}})^T \mathbf{y} = \bar{\mathbf{x}}^T \bar{A}^T \mathbf{y} = \bar{\mathbf{x}}^T A^T \mathbf{y} = -\bar{\mathbf{x}}^T (A\mathbf{y}) = -\mathbf{x} \cdot A\mathbf{y}.$$

   The fact that $\bar{A}^T = A^T$ follows from the fact that all the entries of A are real. This finishes step 1. Plugging in $\mathbf{y} = \mathbf{x}$, where $A\mathbf{x} = \lambda\mathbf{x}$, the left side gives

$$A\mathbf{x} \cdot \mathbf{y} = A\mathbf{x} \cdot \mathbf{x} = \lambda\mathbf{x} \cdot \mathbf{x} = (\overline{\lambda\mathbf{x}})^T \mathbf{x} = \bar{\lambda}\bar{\mathbf{x}}^T \mathbf{x} = \bar{\lambda} \left\|\mathbf{x}\right\|^2.$$

   Plugging it in to the right sides gives

$$-\mathbf{x} \cdot A\mathbf{x} = -\mathbf{x} \cdot \lambda\mathbf{x} = -\bar{\mathbf{x}}^T (\lambda\mathbf{x}) = -\lambda \left\|\mathbf{x}\right\|^2.$$

   For a complex number $\lambda = a + bi$, the only way we can have $\bar{\lambda} = -\lambda$ is if $a = 0$. So all the eigenvalues must be purely imaginary.

4. *Suppose that A is a symmetric matrix, and all of its eigenvalues are equal. Prove that A is a multiple of the identity matrix.*

   Since A is symmetric, it must be diagonalizable: $A = QDQ^T$. Since all the eigenvalues are equal, D is just λI. But $Q(\lambda I) = \lambda Q$, so $A = \lambda QQ^T = \lambda I$.

5. *Let A be the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.*

   (a) *Explain why $\mathbf{x}$ is positive-definite by writing $\mathbf{x}^T A\mathbf{x}$ as a sum of two squares, where $\mathbf{x} = (x, y)$.*
   Here we have $\mathbf{x}^T A\mathbf{x} = 2x^2 + 2y^2 + 2xy = 2(x^2 + y^2 + xy) = 2\left((x + y/2)^2 + 3/4\, y^2\right)$. This is evidently positive.

   (b) *Sketch the ellipse defined by $\mathbf{x}^T A\mathbf{x} = 1$.*
   The eigenvalues in this instance are 1 and 3. The "lined-up" ellipse is $\mathbf{X}^T \Lambda \mathbf{X} = 1$, which is $x^2 + 3y^2 = 1$. This meets the axes at $(1, 0)$ and $(0, 1/\sqrt{3})$.
   But $\mathbf{x}^T A\mathbf{x} = (Q\mathbf{x})^T \Lambda (Q\mathbf{x})$. We can get the ellipse by applying Q to the lined-up ellipse. The eigenvectors are $(1, -1)$ (for $\lambda = 1$) and $(1, 1)$ (for $\lambda = 3$), so $Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The new extreme points will be $(1, -1)$ and $(1/\sqrt{3}, 1/\sqrt{3})$.

6. *Let* $A = \begin{pmatrix} x & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & x \end{pmatrix}$. *For what value(s) of x is this matrix diagonalizable by an orthogonal matrix? for what value(s) of x is it positive definite?*

   It's always symmetric, so always orthogonally diagonalizable.

   For positive definiteness, we have a couple ways to investigate. The easiest is to check that all the upper-left blocks have positive determinant. For use, this gives three conditions: $x > 0$, $x - 4 > 0$, and $x^2 - 4x - 1 > 0$. The last of these holds as long as $x < 2 - \sqrt{5}$ or $x > 2 + \sqrt{5}$. All three will be satisfied as long as $x > 2 + \sqrt{5}$, since this number is a little bigger than 4.

7. *Two masses are arranged in a "fixed-fixed" spring system, with spring constants* 1, 2, *and* 3. *Write down the equation for the masses to be in equilibrium. If they start out of equilibrium, what equation governs their motion? Do we know a strategy to solve an equation of this form?*

   We know that the stiffness matrix is $K = A_0^T C_0 A_0$, where $A_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}$, and $C_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. I didn't actually work out what this is, but you can if you want. Now solving $K\mathbf{u} = \mathbf{f}$ for $\mathbf{u}$, where $\mathbf{f}$ is gravity, will tell you the displacements of the masses when everything is at equilibrium.

   The differential equation is $Md^2\mathbf{u}/dt^2 + K\mathbf{u} = \mathbf{f}$. This sort of thing is amenable to the methods that we looked at earlier. If we introduce new variables $v_i = du_i/dt$, this is $2n$ linear differential equations, and we can find the solutions via our usual method.