Math 18.06, r3/5
Problems #11
May 14, 2013

1. *Suppose* $\lambda_1 = 2$ *and* $\lambda_2 = 5$ *and* $x_1 = (1,1,1)$ *and* $x_2 = (1,-2,1)$. *Choose* λ_3 *and* x_3 *so that* A *is symmetric positive semidefinite but not positive definite.*

   If we want A to be symmetric, the third eigenvector x_3 had better be orthogonal to the other two. The quick way to find a vector orthogonal to two given ones in $\mathbb{R}^3$ is via cross product: $x_3 = x_1 \times x_2 = (3, 0, -3)$.

   Alternately, you can use elimination: x_3 should be in the nullspace of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

   You could also just notice the first and last entries match and guess the answer from that. Either way, x_3 should be a multiple of $(1, 0, -1)$.

   As for the eigenvalue, to get a matrix that's positive semidefinite but not positive definite, we need to use $\lambda_3 = 0$.

   It doesn't actually ask you to compute A, but here's one that works:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 & 3 \\ -2 & 8 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

2. *Suppose:* *A is positive definite symmetric*
   Q is orthogonal (same size as A)
   B is $Q^TAQ = Q^{-1}AQ$.

   Show that: *B is also symmetric.*
   B is also positive definite.

   First we show that B is symmetric. This means we need to check $B^T = B$. Using what we're told,

$$B^T = (Q^TAQ)^T = Q^TA^T(Q^T)^T = Q^TAQ = B,$$

   Note that in the next-to-last step we used the fact that A itself is symmetric ($A^T = A$).

   For positive definiteness, one way is to use the energy test. If x is any nonzero vector, then

$$x^TBx = x^T(Q^TAQ)x = (Qx)^TA(Qx) = y^TAy,$$

   where $y = Qx$. We know that y is nonzero, because Q is orthogonal and therefore has no nullspace.

   Another approach is via eigenvalues. We know that $B = Q^{-1}AQ$, so B is similar to A. That means that they have the same eigenvalues. Since A is positive definite, its eigenvalues are all positive, so those of B are as well.

   A third approach: A is positive definite, so $A = R^TR$ for some R with independent columns. Then $B = Q^TR^TRQ = (RQ)^T(RQ)$. RQ is a matrix with independent columns, since Q is orthogonal. So B is positive definite.

3. *Which of the following are linear transformations? Why or why not?*

   (a) $T : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $T(\mathbf{v}) = |\mathbf{v}|$
   Not: $T((1,0,0)+(-1,0,0)) = T(0) = 0$, while $T((1,0,0)) + T((-1,0,0)) = 1+1 = 2$.

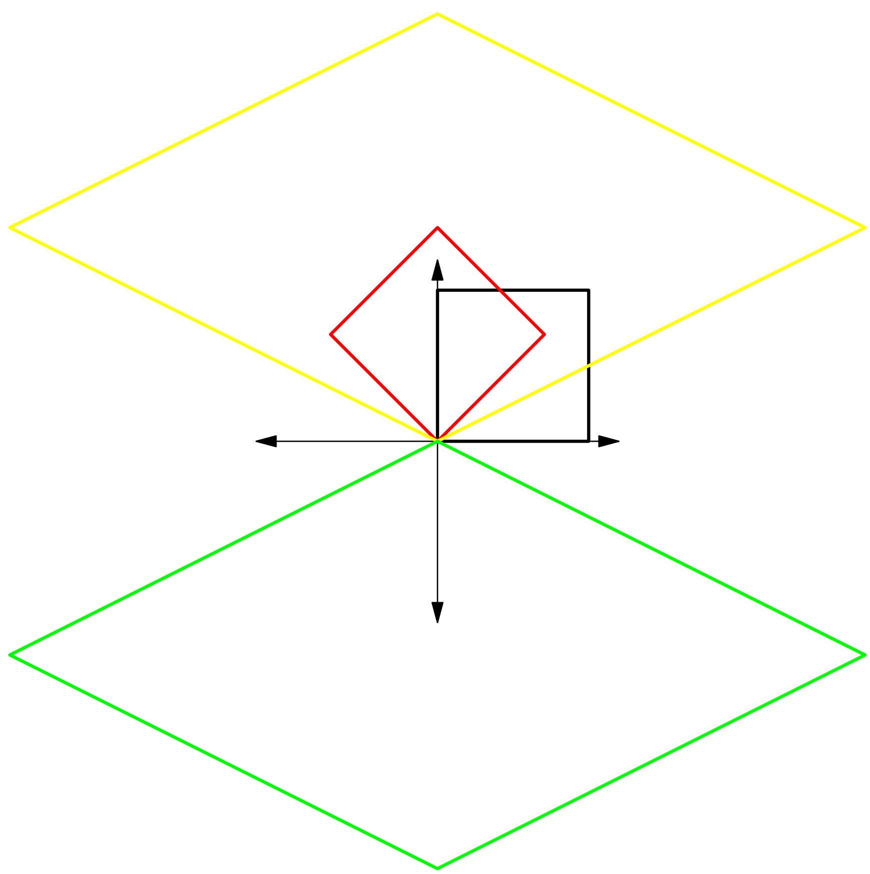
   (b) $T(\mathbf{v}) =$ *largest component of* $\mathbf{v}$
   Not: $T((1,2,3) + T((3,2,1)) = T((4,4,4)) = 4$, while $T((1,2,3)) + T((3,2,1)) = 3+3 = 6$.

   (c) $T(a,b,c) = (b,c,a)$.
   This one is.

4. *Consider the matrix with SVD*

$$A = U\Sigma V^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

   Sketch the image of the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, *and* $(1,1)$ *under* A *by applying each of the three matrices of the SVD.*



   The original square S is black, then, red, yellow, and green for $V^T S$, $\Sigma V^T S$, and $U\Sigma V^T S$.

5. *Consider the vector space of functions spanned by* $\sin x$, $\cos x$, $\sin 2x$, *and* $\cos 2x$.

   (a) *Write down the matrices* ∂ *and* $\int$ *for differentiation and integration on this space.*

Computing the derivatives and integrals of these five functions, we get

$$\partial = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \qquad \int = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

We have $\partial \int(f) = f$ and $\int \partial(f) = f$.

(b) *What are $\partial \int$ and $\int \partial$?*

In this case, both are the identity.

(c) *How can you find the matrix for taking the second derivative?*

We can square the matrix for D! This gives

$$D^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

We could also get this answer by directly computing the second derivative of each of the basis vectors.

(d) *How would your answer change if we added the function 1 to this basis?*

Integration would take us out of this space of functions: the integral of the function 1 is x, which isn't periodic at all. So we can't really write down a matrix for integration.

6. *Suppose that f has period 2π and $f(-x) = -f(x)$ for all x (i.e. x is an odd function). What does this tell you about its Fourier coefficients?*

Recall the formulas for the coefficients:

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

If f is odd, then $f(x)\cos kx$ is odd times even, which is odd. Since f has period 2π, the integral from 0 to 2π is the same as that from $-\pi$ to π, which is 0 because the integrand is odd. So all of the a_k's must be odd in this case.

7. *Why are the eigenvalues of any Hermitian matrix real?*

We know $\mathbf{x}^H A\mathbf{x}$ is a real number if A is hermitian and $\mathbf{x}$ is any vector. Now suppose $A\mathbf{x} = \lambda\mathbf{x}$, and so $\mathbf{x}^H A\mathbf{x} = \mathbf{x}^H \lambda\mathbf{x} = \lambda(\mathbf{x}^H\mathbf{x})$. This gives $\lambda = (\mathbf{x}^H A\mathbf{x})/(\mathbf{x}^H\mathbf{x})$, a quotient of two real numbers. So λ is real.

8. *What class of matrices does P belong to? (invertible, Hermitian, unitary)*

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}$$

What are the eigenvalues of P?

This matrix is invertible, since the determinant is $i^3 = -i \neq 0$. Unitary is the complex analog of orthogonal – and indeed, it's clear that for this matrix the columns are orthogonal with respect to the complex inner product. So this is unitary. It's not hermitian; remember that means equal to its own conjugate transpose.