Usually I'd prepare a review sheet before an exam like this, but the book already does a better job than I would have – I strongly recommend making sure you understand the *Review of the Key Ideas* at the end of each section.

The next page has a few practice problems you might find useful. A warning: I probably spent too long on computation problems in recitation, so you won't find any here. There will almost certainly be some on the exam, so do some problems from old tests or the recitation sheets!

Here are some computations you should be ready for (see recitation problems for examples):

- Decompositions: $A = LU$, $PA = LU$, $A = LDU$, $A = LDL^T$ (if A symmetric).
- Inverse of a square matrix.
- rref form of a matrix (and the various things you can read off of this – the rank, the nullspace, the pivot columns).
- All solutions to $A\mathbf{x} = \mathbf{0}$ (i.e., the nullspace of A; use the "special solutions" from free columns).
- All solutions to $A\mathbf{x} = \mathbf{b}$ (find a particular solution via elimination, putting 0s in free vars).

Math 18.06, r3/5
Problems #Review 1
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1. *Let A and B be the matrices*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

Does there exist a vector $\mathbf{b}$ *such that* $A\mathbf{x} = \mathbf{b}$ *has a unique solution? Such that* $B\mathbf{x} = \mathbf{b}$ *has a unique solution?*

For $A\mathbf{x} = \mathbf{b}$, no: the nullspace of A is not zero, and given any $\mathbf{b}$, if $A\mathbf{x} = \mathbf{b}$ is one solution, then $\mathbf{x} + v_{\mathbf{n}}$ is another solution for any $v_{\mathbf{n}}$ in the nullspace. On the other hand, B has no nullspace, so if $B\mathbf{x} = \mathbf{b}$ has a solution, it's automatically unique.

2. *Give an example of a matrix A such that*

- *For some vector* $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ *has no solutions.*
- *For any* $\mathbf{b}$ *for which* $A\mathbf{x} = \mathbf{b}$ *has any solutions, it has infinitely many.*

What matrices have this property?

One such is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To get the first property, it must be that the column space of A is not all of $\mathbb{R}^m$ (where A is an $m \times n$ matrix). To get the second, A must have a nonzero vector in the nullspace (equivalently, a dependent column). Any matrix with these two properties will do the trick.

3. *Suppose that an LU decomposition for A has* $L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. *What row operations must we have made when going from A to U by elimination?*

This matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = E_{12}^{-1} E_{13}^{-1}.$$

Thus $E_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $E_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. The first operation was to add three times the first row to the second, and the second was to subtract the first row from the third.

4. *Suppose that A is a 3×5 matrix of rank 2. If B is a 5×10 matrix, what are the possible ranks of AB? If C is 10×3, what are the possible ranks of CA?*

The columns of AB are linear combinations of the columns of A, so the column space has dimension at most 2. The rows of CA are linear combinations of the rows of A, so the row space has dimension at most 2. In both cases the rank could also be 0 or 1.

5. *Give an example of a matrix whose entries are all 0s or 1s, but which has a pivot that's something other than 1 or −1.*

There are lots of ways to do this, though your matrix will need to be at least 3×3. One example is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

6. *For which 3×3 matrices A does rref take the form $R = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$? (Describe conditions on the columns of A)*

The second column isn't a pivot column, so it must be a multiple of the first column. The third column is, so it must be linearly independent of the first two. In other words, the first column can be anything, while the second column

7. *The inverse of an upper-triangular matrix is upper-triangular. Why?*

8. (a) *An upper triangular matrix is invertible if ____________________.*

   An upper triangular matrix is invertible if <u>it has no 0 on the diagonal</u>.

   (b) *Let $\mathbf{v} = (1, 2, 3)$ be a three-dimensional vector. The set of 3×3 matrices with ($\mathbf{v}$ in the column space/$\mathbf{v}$ in the nullspace/both/neither) is a subspace of the space of 3×3 matrices.*

   The set of matrices with $\mathbf{v}$ in the column space isn't a subspace: any subspace must contain the zero 3×3 matrix, but this matrix doesn't have $\mathbf{v}$ in its column space.

   The set of matrices with $\mathbf{v}$ in the nullspace is a vector space. If $A\mathbf{v} = \mathbf{0}$ and $B\mathbf{v} = \mathbf{0}$, then $(A + B)\mathbf{v} = A\mathbf{v} + B\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. This means that the space is closed under addition. We also need to check it's closed under scalar multiplication: if c is a number, then $(cA)\mathbf{v} = c(A\mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$.

   (c) *The number of $n \times n$ permutation matrices with a "1" in the upper left is ___.*

   The first row contains no other 1s, and the first column contains no other 1s. The rest of the matrix has to be an $n - 1 \times n - 1$ permutation matrix, and there are $(n - 1)!$ of these.

One way to see this is via Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & d & e & 0 & 1 & 0 \\ 0 & 0 & f & 0 & 0 & 1 \end{array}\right]$$

It's already upper-triangular, so we only need to do the back-substitution steps (and dividing rows by numbers). Subtracting a multiple of a row from a row above it can only change the above-diagonal entries of the matrix on the right, and so at the end of the process we'll get an upper-triangular inverse.

9. *Suppose A is a 3×3 matrix which isn't invertible. Is it possible that there is 3×3 matrix B such that AB is invertible? Can AB be invertible if B has other dimensions?*

If B is square, AB can't be invertible. The columns of AB are linear combinations of the columns of A, so the column space of AB has dimension at most equal to the dimension of the column space of A, which is less than 3. But an invertible matrix has a 3-dimensional column space.

If B is smaller, AB can be invertible. AB will still have rank at most 2, but if AB is a 2×2 matrix this isn't a problem. Here's an example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

10. *Let V be the subspace of $\mathbb{R}^3$ consisting of vectors (x, y, z) for which $x + 2y + 3z = 0$. What is the dimension of V? Give a basis for V.*

    Here's one approach that uses the methods we've covered. Let A be the 1×3 matrix $(1\ 2\ 3)$. The nullspace of A is exactly the subspace V, and we know how to find a basis for a nullspace.

    A is already in rref form, with the second and third columns free, and the special solutions are $(-2, 1, 0)$ and $(-3, 0, 1)$. These give a basis for V.

11. *Let V be the vector space of polynomials of degree less than or equal to 2 (i.e. functions $f(x) = ax^2 + bx + c$). What is a basis for V? Do the polynomials with $f(1) = 0$ form a subspace? Those with $f(1) \geq 0$?*

    A basis is given by the three functions 1, x, and x^2. The polynomials with $f(1) = 0$ are a subspace, but those with $f(1) \geq 0$ aren't: $f(x) = x$ is such a polynomial, but multiplying by -1 we get $-x$, which isn't.