Math 121 (Lesieutre); §5.3; September 11, 2017

1. *What is an exponential function?*

It's a function of the form Ca^x, where a is a positive real number, and C is a nonzero constant. Here's an example: $7 \cdot 2^x$. Or $\frac{355}{113}\pi^x$.

2. *Sketch the graphs of the following exponential functions by first making a table of values.*

$$f(x) = 2^x, \qquad g(x) = \left(\frac{1}{2}\right)^x, \qquad h(x) = -2^x.$$

Here are the tables of values...

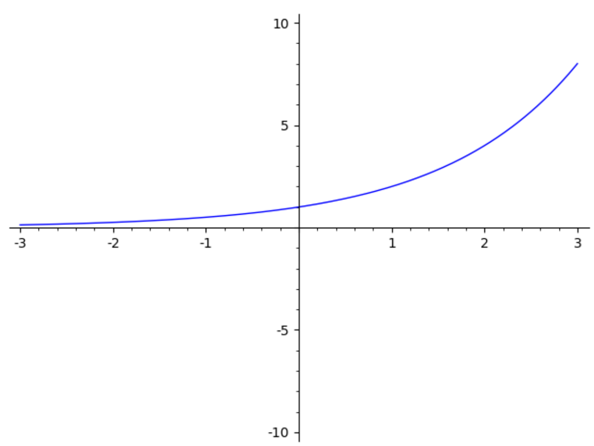
x	2^x
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

x	$(1/2)^x$
-3	8
-2	4
-1	2
0	1
1	$1/2$
2	$1/4$
3	$1/8$

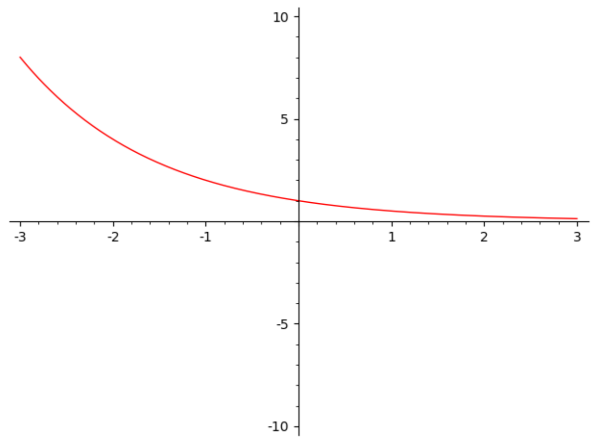
x	-2^x
-3	$-1/8$
-2	$-1/4$
-1	$-1/2$
0	-1
1	-2
2	-4
3	-8

...and here are the plots.

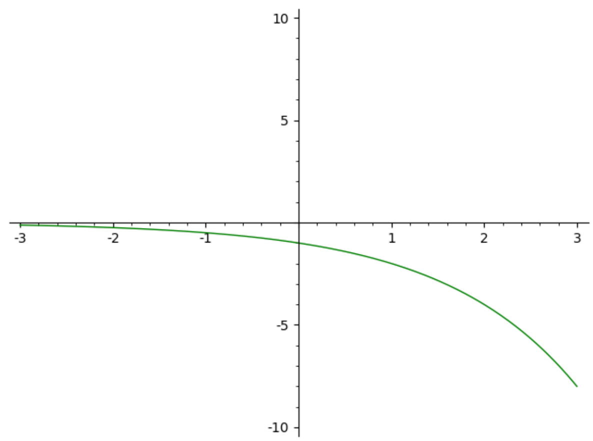
2^x:



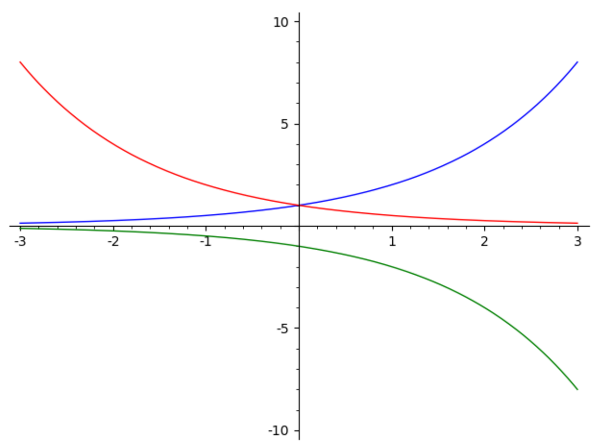
$(1/2)^x$ (aka 2^{-x}):



-2^x:



For comparison, here they are all together:

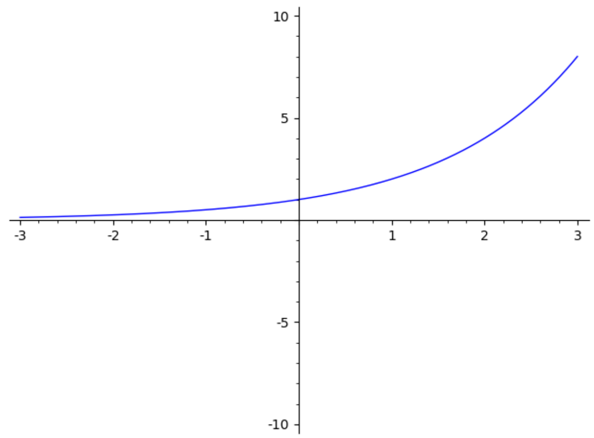


3. *Using transformations of graphs, graph the following function:*

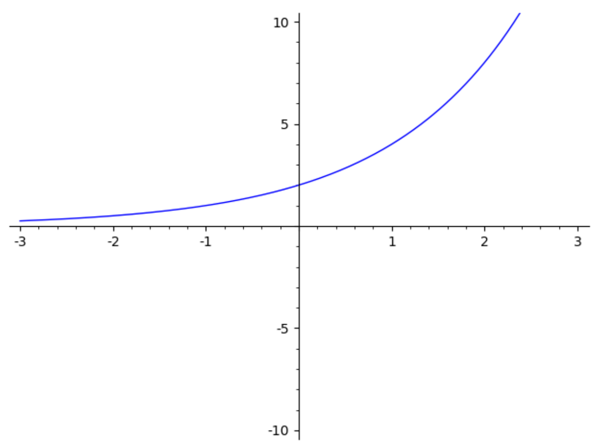
$$F(x) = 3 - 2^{x+1}$$

We start with 2^x. That you already know. 2^{x+1} shifts it right by 1. -2^{x+1} flips that over the axis. $(3 - 2^{x+1}$ shifts it up by 3. Here it is, step-by-step:

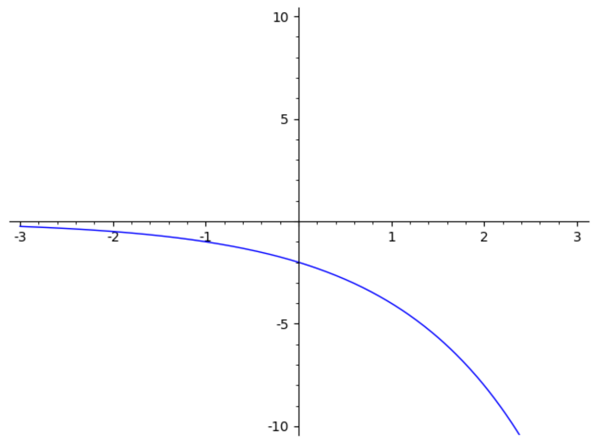
2^x:



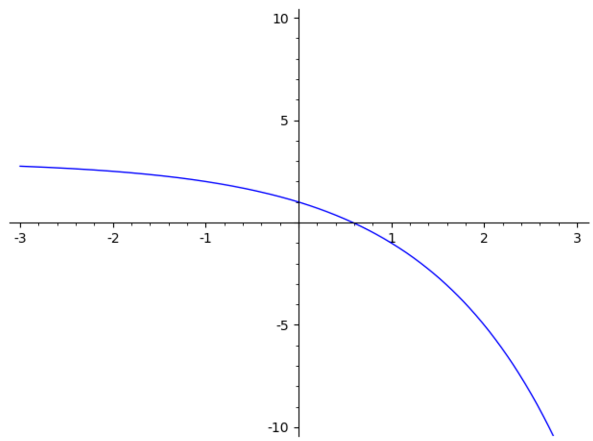
2^{x+1}:



-2^{x+1}:



$3 - 2^{x+1}$:



4. *What is e? Two ways to define it:*

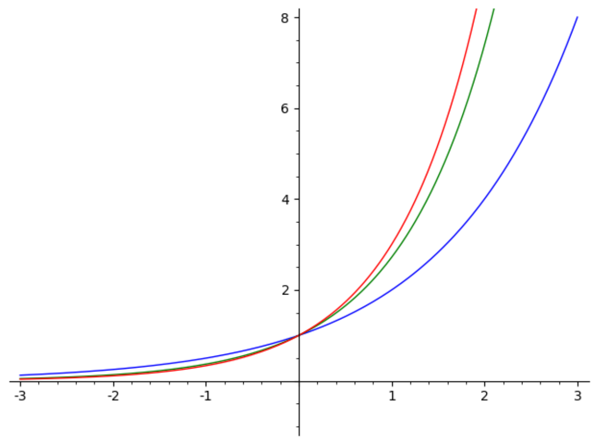
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad e = \sum_{j=0}^{\infty}\frac{1}{j!} = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots$$

n	$\left(1+\frac{1}{n}\right)^n$
1	2.0000000
10	2.5937425
100	2.7048138
1000	2.7169239
10000	2.7181459
100000	2.7182682
1000000	2.7182805
10000000	2.7182817

n	$\sum_{j=0}^{n}\frac{1}{j!}$
0	1.0000000
1	2.0000000
2	2.5000000
3	2.6666667
4	2.7083333
5	2.7166667
6	2.7180556
7	2.7182540

5. *e is in many ways a number like π. It's irrational, so we express it using a symbol: however, don't forget that it isn't a variable! We can still graph exponential functions with e as the base.*

Here are the graphs of $f(x) = 2^x$, $g(x) = e^x$, and $h(x) = 3^x$. Can you tell which is which?



Sketch the graph of $y = e^{-x}$ on the same axes.

The fastest-growing one (red) is 3^x, and the slowest-growing (blue) is 2^x. e is between 2 and 3, so unsurprisingly, the graph comes in somewhere in the middle (it's the green one).

6. *How different are exponential and linear growth? Let's check by making a table of values for the following functions:*

$$f(x) = 2x + 1, \qquad g(x) = 3^x$$

Here are the tables:

x	$2x+1$
-5	-9
-4	-7
-3	-5
-2	-3
-1	-1
0	1
1	3
2	5
3	7
4	9
5	11

x	3^x
-5	1/243
-4	1/81
-3	1/27
-2	1/9
-1	1/3
0	1
1	3
2	9
3	27
4	81
5	243

The exponential function grows much much faster than the linear one, when x is a big positive number. For small x, the exponential is very close to 0.

7. *Now let's consider some exponential equations, and how to solve them.*

$$2^{2x} = 16, \quad 3^{x+4} = \frac{1}{27}, \quad 8^{-x+14} = 16^x, \quad e^{x^2} = e^{3x} \cdot e^{-2}.$$

Notice that $2^4 = 16$, so we need to have $2x = 4$. That means $x = 2$.
For the next, $3^3 = 27$, so $3^{-3} = \frac{1}{27}$. Our solution will happen when $x + 4 = -3$, so $x = -7$.
For the third one,

$$\begin{aligned} 8^{-x+14} &= 16^x \\ (2^3)^{-x+14} &= (2^4)^x \\ 2^{3(-x+14)} &= 2^{4x} \\ 2^{-3x+42} &= 2^{4x} \\ -3x + 42 &= 4x \\ 7x &= 42 \\ x &= 6. \end{aligned}$$

At last,

$$\begin{aligned} e^{x^2} &= e^{3x} \cdot e^{-2} \\ e^{x^2} &= e^{3x-2} \\ x^2 &= 3x - 2 \\ x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0. \end{aligned}$$

So $x = 1$ or $x = 2$.