Math 210 (Lesieutre)
Exam review #2
April 28, 2017

Problem 1. *Consider the integral* $\int_{x=0}^{3}\int_{y=0}^{x^3} xy\,dy\,dx$. *Sketch the region of integration, and reverse the order of the integrals.*

It's the region underneath the graph of $y = x^3$ and above the x-axis, for x between 0 and 3. The top right corner of the region is the point $(3, 27)$, and so on this region we're going to have y from 0 to 27. For a given y, the lower bound on x is the graph, and the upper bound is $x = 0$. The graph is $y = x^3$, which means $x = \sqrt[3]{y}$. So our integral is going to be:

$$\int_{y=0}^{27}\int_{x=\sqrt[3]{y}}^{3} xy\,dx\,dy.$$

Problem 2. *Compute the integral* $\oint_C \mathbf{F}\cdot d\mathbf{n}\,ds$ *where* $\mathbf{F} = \langle x^2, y\rangle$ *and* C *is a triangular path from* $(1,0)$ *to* $(0,2)$ *to* $(0,0)$ *and back to* $(1,0)$.

This is a problem for the normal form Green's theorem, which says that

$$\oint_C \mathbf{F}\cdot d\mathbf{r}\,ds = \iint_R \operatorname{div}\mathbf{F}\,dA.$$

Here $\operatorname{div}\mathbf{F} = 2x + 1$, and the bounds for R with be $0 \le x \le 1$ and $0 \le y \le 2 - 2x$. So we want

$$\int_{x=0}^{1}\int_{y=0}^{2-2x} 2x + 1\,dydx$$

Inner:

$$\int_{y=0}^{2-2x} 2x + 1\,dy = (2 - 2x)(2x + 1) = -4x^2 + 2x + 2$$

Outer:

$$\int_{x=0}^{1} -4x^2 + 2x + 2 = -\frac{4x^3}{3} + x^2 + 2x\Big|_0^1 = -\frac{4}{3} + 1 + 2 = \frac{5}{3}.$$

Problem 3. a) *What is the tangent plane to the surface* $x^2 + 2y^2 + 2z^2 = 8$ *at the point* $(2,1,1)$*?*

The tangent plane is normal to the gradient of the function defining the surface: $f(x, y, z) = x^2 + 2y^2 + 2z^2 - 8$. The gradient is $\nabla f = \langle 2x, 4y, 4z\rangle$, which at the point in question is $\nabla f(2, 1, 1) = \langle 4, 4, 4\rangle$. So the plane is normal to this vector. Since the tangent plane passes through the point $(2, 1, 1)$, it must be

$$4(x - 2) + 4(y - 1) + 4(z - 1) = 0.$$

b) *Consider the function* $f(x,y) = 3 + x^2\sqrt{y}$*. Use a linear approximation centered at* $(2,1)$ *to approximate the value of* $f(2.1, 1.2)$.
We have

$$f_x = 2x\sqrt{y}$$
$$f_y = \frac{x^2}{2\sqrt{y}}.$$

At the point $(2,1)$, these things evaluate to

$$f(2,1) = 7$$
$$f_x(2,1) = 4$$
$$f_y(2,1) = 2$$

The formula for linear approximation says that

$$\begin{aligned} f(x,y) &\approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 7 + 4(x-2) + 2(y-1). \end{aligned}$$

Plugging in $(x,y) = (2.1, 1.2)$ as requested, we get

$$f(2.1, 1.2) \approx 7 + 4(x-2) + 2(y-1) = 7 + 4(0.1) + 2(0.2) = 7.8.$$

For comparison, the actual value is 7.831.

Problem 4. a) *Compute the gradient field of the function* $\phi(x,y,z) = x^2 - yz$*. What kind of field do you get?*
The gradient is

$$\nabla\phi = \langle 2x, -z, -y \rangle.$$

It's a conservative field.

b) *For your vector field from part (a), compute* $\int_C \mathbf{F} \cdot d\mathbf{r}$*, where* C *is parametrized by* $\mathbf{r}(t) = \langle t, t^2, t\rangle$ *with* $1 \le t \le 3$.
It's a conservative field, so we can do this using the fundamental theorem for line integrals. The beginning of our path is $\mathbf{r}(1) = \langle 1,1,1\rangle$, and the end is $\mathbf{r}(3) = \langle 3,9,3\rangle$. The fundamental theorem then tells us that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(3,9,3) - \phi(1,1,1) = (9-27) - (1-1) = -18.$$

Problem 5. a) *Find the flux of* $\mathbf{F} = \langle y, x, x+z\rangle$ *across the top face of the unit cube, with an upward normal vector.*

Parametrize it $\mathbf{r}(u,v) = \langle u, v, 1\rangle$. Then $\mathbf{t}_u = \langle 1,0,0\rangle$ and $\mathbf{t}_v = \langle 0,1,0\rangle$, so $\mathbf{t}_u \times \mathbf{t}_v = \langle 0,0,1\rangle$. The flux is

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_{u=0}^{1}\int_{v=0}^{1} \langle v, u, 1+u\rangle \cdot \langle 0,0,1\rangle \; dv\,du \\ &= \int_{u=0}^{1}\int_{v=0}^{1} 1+u\,dv\,du = \int_{u=0}^{1} 1+u\,du = u + \frac{u^2}{2}\Big|_0^1 = \frac{3}{2}.\end{aligned}$$

b) *Find the outward flux of* $\mathbf{F} = \langle y, x, x+z\rangle$ *across the entire unit cube.*

This one we can do using the divergence theorem. The divergence is $\nabla\cdot\mathbf{F} = 0+0+1 = 1$, and so

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_D \nabla\cdot\mathbf{F}\,dV \\ &= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 1 dz\,dy\,dx = 1.\end{aligned}$$

Problem 6. *Consider the function* $f(x,y) = x + 2y + e^{xy}$*. Find the direction of steepest ascent at* $(x,y) = (0,0)$*, and the rate of ascent in that direction.*

The direction of steepest ascent is always the direction of the gradient, which is $\nabla f = \langle 1 + ye^{xy}, 2 + xe^{xy}\rangle$, so that $\nabla f(0,0) = \langle 1, 2\rangle$. That's the direction of steepest ascent. The rate of ascent is the length of the gradient, which is $\sqrt{1^2+2^2} = \sqrt{5}$.

Problem 7. *Find the area of a tringle with vertices at* $(1,1,1)$*,* $(1,2,1)$*, and* $(1,2,3)$*.*

Let's call the vertices A, B and C respectively. Then $AB = \langle 0,1,0\rangle$ and $AC = \langle 0,1,2\rangle$. The area of the triangle is half the area of the parallelogram with these two vectors as legs, which is given by the cross product of these vectors. We have

$$\langle 0,1,0\rangle \times \langle 0,1,2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \langle 2,0,0\rangle\,.$$

The length of this vector is 2, and so the area of the triangle is 1.

Problem 8. *Suppose that a particle has* $\mathbf{v}(t) = \langle -\sin t, \cos t, 2\rangle$*, and initial position* $\mathbf{r}(0) = \langle 1,2,3\rangle$*. Find a formula for* $\mathbf{r}(t)$*.*

We know that $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt + \mathbf{C}$. The indefinite integral is $\langle \cos t, \sin t, 2t\rangle$. So the question is what vector to use for $\mathbf{C}$.

The answer is that we should plug in 0 to figure this out. So far we know that $\mathbf{r}(t) = \langle \cos t, \sin t, 2t\rangle + \mathbf{C}$ (here $\mathbf{C}$ is an honest constant – no t's). Then $\mathbf{r}(0) = \langle 1,0,0\rangle + \langle C_1, C_2, C_3\rangle = \langle 1,2,3\rangle$, and so $\mathbf{C} = \langle 0,2,3\rangle$. Our answer is then

$$\mathbf{r}(t) = \langle \cos t, \sin t + t, 2t + 3\rangle\,.$$