Math 210 (Lesieutre)
11.2: Vectors in three dimensions
January 11, 2017

Problem 1. *Consider the two points in three dimensions with coordinates $P = (1, 2, 3)$ and $Q = (-1, 2, 1)$.*

a) *What is $\overrightarrow{PQ}$? What is $\overrightarrow{QP}$? How do these differ?*
$\overrightarrow{PQ}$ is given by $(-1, 2, 1) - (1, 2, 3) = \langle -2, 0, -2 \rangle$. $\overrightarrow{QP}$ is given by $(1, 2, 3) - (-1, 2, 1) = \langle 2, 0, 2 \rangle$. These differ by a factor of -1: they have the same length, but point in opposite directions.

b) *What is the length of $\overrightarrow{PQ}$?*
It's given by $\sqrt{(-2)^2 + 0^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$.

c) *Find a vector of length 3 parallel to $\overrightarrow{PQ}$.*
We want $3\overrightarrow{PQ}/\left|\overrightarrow{PQ}\right| = \left\langle \frac{-6}{2\sqrt{2}}, 0, -\frac{-6}{2\sqrt{2}} \right\rangle = \left\langle -\frac{3\sqrt{2}}{2}, 0, -\frac{3\sqrt{2}}{2} \right\rangle$.

d) *What is the midpoint of the line segment between P and Q?*
The midpoint formula says it's $((1 + (-1))/2, (2 + 2)/2, (3 + 1)/2) = (0, 2, 2)$.

Problem 2. *Describe the sphere with equation $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 25$.*

The sphere has center $(1, -2, 3)$ and radius 5.

Problem 3. *Relative to the air, an airplane is flying 30 degrees west of north, with speed 500 MPH. The wind is traveling due north at 100 MPH. What is the velocity vector of the airplane relative to the ground?*

First we need to translate to find the components of the airplane vector. In polar coordinates, our vector has angle $90 + 30 = 120$. The x-component is $500 \cos 120 = 500(-1/2) = -250$. The y-component is $500 \sin 120 = 500(\sqrt{3}/2) = 250\sqrt{3}$. So the velocity relative to the air is $\left\langle -250, 250\sqrt{3} \right\rangle$. The velocity of the air relative to the ground is $\langle 0, 100 \rangle$.

Let me write $\mathbf{v}_{X/Y}$ for the speed of X relative to Y. As we saw in class,

$$\mathbf{v}_{\text{plane/ground}} = \mathbf{v}_{\text{plane/air}} + \mathbf{v}_{\text{air/ground}} = \left\langle -250, 250\sqrt{3} \right\rangle + \langle 0, 100 \rangle = \left\langle -250, 250\sqrt{3} - 100 \right\rangle.$$

Problem 4. *A 10-pound weight is suspended from two strings, each making a 45 degree angle with the ceiling. How much force is exerted on the mass by each of the strings?*

This is a classic sort of problem. Since the situation is symmetric, the two forces from the strings are equal in magnitude, but in different directions. Let's say M is the answer. The two force vectors from the strings are

$$\mathbf{F}_1 = \left\langle -M\frac{\sqrt{2}}{2}, M\frac{\sqrt{2}}{2} \right\rangle$$
$$\mathbf{F}_2 = \left\langle M\frac{\sqrt{2}}{2}, M\frac{\sqrt{2}}{2} \right\rangle$$

(the first of those is $M \cos 45°$ for the left string, etc; similar to the previous problem)

The gravitational force is $\mathbf{g} = \langle 0, -10 \rangle$: a force of 10 directly downward.

Adding these up we should get 0: if the object isn't moving, the forces have to balance out:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{g} = \mathbf{0}.$$

So $\left\langle 0, \sqrt{2}M \right\rangle + \langle 0, -10 \rangle = \langle 0, 0 \rangle$, which means $\sqrt{2}M = 10$, so $M = 10/\sqrt{2} = 5\sqrt{2}$. (This tells use how strong each string needs to be to keep the mass from falling: each individually is holding the same force as if it were supporting a single $5\sqrt{2} \approx 7.07$.

Problem 5. *Let R be a parallelogram with legs given by the vectors* $\mathbf{u}$ *and* $\mathbf{v}$*. Prove that both diagonals of R have the same midpoint.*

Place $\mathbf{u}$ and $\mathbf{v}$ with their tails at the origin. The diagonal from the origin to the far corner of the parallelogram is $\mathbf{u} + \mathbf{v}$, and so the position vector for the midpoint is $(\mathbf{u} + \mathbf{v})/2$.

The second diagonal (oriented to start at $\mathbf{u}$ and end at $\mathbf{v}$) is the vector $\mathbf{v} - \mathbf{u}$. To get the position vector, we need to add half of this to the position of its tail: $\mathbf{u} + (\mathbf{v} - \mathbf{u})/2 = (\mathbf{u} + \mathbf{v})/2$. So the two answers match.