Math 210 (Lesieutre)
11.6: Calculus of vector valued functions
January 23, 2016

Problem 1. *The position of a planet is given by* $\mathbf{r}(t) = \langle 3\cos t, 2\sin t\rangle$*. Find a unit tangent vector to the path of the planet.*

First we find that

$$\mathbf{r}'(t) = \langle -3\sin t, 2\cos t\rangle .$$

This gives

$$|\mathbf{r}'(t)| = \sqrt{9\sin^2 t + 4\cos^2 t}.$$

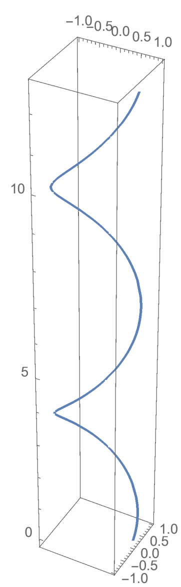
Nothing to be done with that, really, but it gives

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle \frac{3\cos t}{\sqrt{9\sin^2 t + 4\cos^2 t}}, \frac{2\sin t}{\sqrt{9\sin^2 t + 4\cos^2 t}} \right\rangle .$$

Problem 2. *Consider the function* $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$

a) *Sketch* $\mathbf{r}(t)$ *for* $0 \le t \le 4\pi$*.*

This one we did last time. It's the helix sketched below.



b) *Compute* $\mathbf{r}'(t)$ *and* $\mathbf{r}''(t)$*. Plug in* $t = 0$ *and* $t = \pi/2$ *to your answer. Do these make sense? Can you think of any other "sanity checks"?*

The derivatives are given by

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$$
$$\mathbf{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$$

When $t = 0$, we have $\mathbf{r}'(0) = \langle 0, 1, 1\rangle$. This is a vector whose projection to the xy-plane points in the y-direction, but also has z part equal to 1. This seems to match up with the picture. Similarly, we find that $\mathbf{r}''(0) = \langle -1, 0, 1\rangle$, which points in the negative x-direction and upwards. Seems plausible.

You might also notice that $\mathbf{r}''(t) = 0$ for all t. This also makes sense: the rate at which the particle is moving upwards is not changing.

c) *Find a unit tangent vector for the parametrized curve, in terms of* t.

We know that the tangent vector is given by

$$\mathbf{r}'(t) = \langle \sin t, \cos t, 1\rangle,$$

for any t. The problem is that this isn't a unit vector, since

$$|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}.$$

To get a unit vector, we want

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle \frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Problem 3. *Consider the path given by* $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$.

a) *Write down an equation for the tangent line to* $\mathbf{r}(t)$ *at* $t = 1$.

The direction of the line is given by $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. At $t = 1$ this is $\langle 1, 2, 3\rangle$. It goes through the point $\mathbf{r}(1) = \langle 1, 1, 1\rangle$, so the equation is given by

$$\mathbf{x}(t) = \langle 1, 1, 1\rangle + \langle 1, 2, 3\rangle\, t.$$

b) *What is* $\frac{d}{dt}(\mathbf{r}(t) \cdot \langle 1, 2, 3\rangle)$*?*

Let $\mathbf{s}(t) = \langle 1, 1, 3\rangle$. Then $\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) + \mathbf{s}'(t) = \langle 1, 2t, 3t^2\rangle \cdot \langle 1, 2, 3\rangle + \langle t, t^2, t^3\rangle \cdot \langle 0, 0, 0\rangle = 1 + 4t + 9t^2$.

Problem 4. a) *What is*

$$\frac{d}{dt} |\mathbf{x}(t)|^2,$$

in terms of $\mathbf{x}(t)$ *and its derivatives?*

This is

$$\frac{d}{dt}\mathbf{x}(t) \cdot \mathbf{x}(t) = \mathbf{x}(t) \cdot \mathbf{x}'(t) + \mathbf{x}'(t) \cdot \mathbf{x}(t) = 2\mathbf{x}(t) \cdot \mathbf{x}'(t).$$

b) *Check your answer to (a) for* $\mathbf{x}(t) = \langle \cos t, \sin t\rangle$. *Does your answer make sense?*

According to the product rule, it's $\mathbf{x}(t) \cdot \mathbf{x}'(t) + \mathbf{x}'(t) \cdot \mathbf{x}(t) = \langle \cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle + \langle -\sin t, \cos t\rangle \cdot \langle \cos t, \sin t\rangle = 0$.

c) *Suppose that* $\mathbf{x}(t)$ *has constant length. What does this tell you about the relationship between* $\mathbf{x}(t)$ *and* $\mathbf{x}'(t)$*?*

It tells you that they are orthogonal for all values of t!

Problem 5. *Let* $\mathbf{s}(t) = \langle -\sin t, \cos t, 1\rangle$. *Compute* $\int \mathbf{s}(t)\, dt$.

We just integrate each component:

$$\int \langle -\sin t, \cos t, 1\rangle = \langle \cos t, \sin t, t\rangle + \langle c_1, c_2, c_3\rangle .$$

This undoes what we did in the second problem. We'll see next time that if $\mathbf{s}(t)$ is the velocity of a particle, $\mathbf{r}(t)$ is its position.