Math 210 (Lesieutre)
12.1: Planes and surfaces, part 1
January 30, 2017

Problem 1. a) *What is the equation for a plane passing through* $(1, 2, 3)$ *and with normal vector* $\langle 2, -1, 3\rangle$*?*
The equation is $2(x-1)-(y-2)+3(z-3)=0$, which simplifies to $2x-y+3z=9$.

b) *At what point does this plane meet the* x*-axis?*
A point on the x-axis has $y=0$ and $z=0$. So we need $2(x-1)-(0-2)+3(0-3)=0$, which means $2x=9$. So the point is $(9/2, 0, 0)$.

c) *At what point does this plane meet the line* $\mathbf{r}(t)=\langle -t-1, 2t, t+1\rangle$*?*
We need to know for what value of t the equation of the plane is satisfied, which means that $2(-1-t)-2t+3(1+t)=9$. This gives $1-t=9$, so $t=-8$.
We are supposed to find the point where they intersect, rather than the time t. So we should plug this back in to our original equation: $\mathbf{r}(-8)=\langle 7, -16, -7\rangle$. This does indeed satisfy $2x-y+3z=9$

Problem 2. a) *Find the equation of the plane containing the three points* $P=(1,3,1)$*,* $Q=(1,0,3)$*, and* $R=(5,4,-1)$*.*
We need to find the normal vector. It will be perpendicular to both $\vec{PQ}$ and $\vec{PR}$, so we want to take the cross product to find $\mathbf{n}$. We obtain

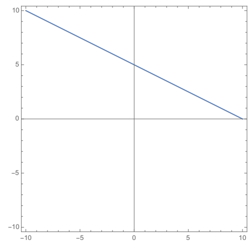
$$\begin{aligned}\vec{PQ} &= (1,0,3)-(1,3,1)=(0,-3,2)\\ \vec{PR} &= (5,4,-1)-(1,3,1)=(4,1,-2)\\ \vec{pq}\times\vec{pr} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -3 & 2\\ 1 & 1 & -2\end{vmatrix} = 4\mathbf{i}+8\mathbf{j}+12\mathbf{k}=\langle 4,8,12\rangle\,.\end{aligned}$$

That's our normal vector. To get the equation, we just need to plug in a point it passes through. We might as well use P. You could also use Q or R; you'll get the same answer for the plane at the end of the day.
So the equation is

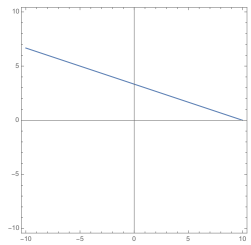
$$\begin{aligned}a(x-x_0)+b(y-y_0)+c(z-z_0)&=0\\ 4(x-1)+8(y-3)+12(z-1)&=0\\ 4x+8y+12z&=40\\ x+2y+3z&=10.\end{aligned}$$

b) *Draw a sketch of the plane by computing its traces in the three coordinate planes.*
In the xy-plane, we plug in $z=0$ and get $4x+2y=13$. In the xz-plane, we plug in $y=0$ and get $4x+3z=13$. In the yz-plane, we plug in $x=0$ and get $2y+3z=13$.
These three planes are plotted below.

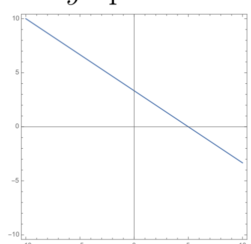
xy-plane:



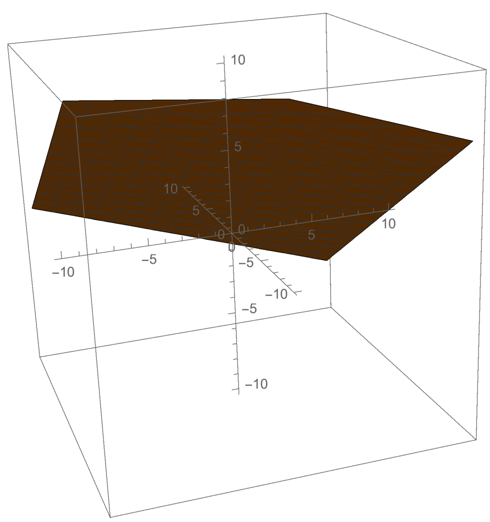
xz-plane:



yz-plane:



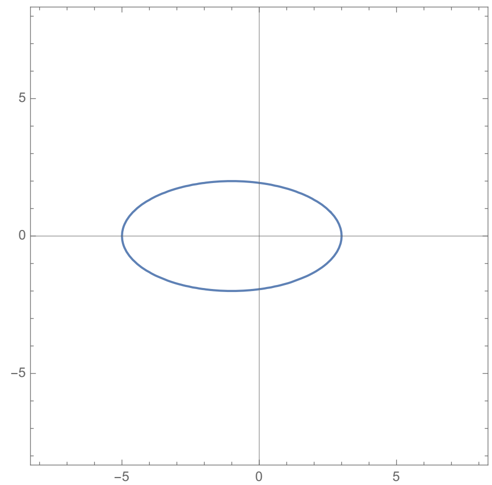
Putting it all together, we get the following plane. I'm drawing a "back view" that makes it easier to see the axes.



Problem 3. *Graph the cylinder* $x^2 + 2x + 4z^2 = 15.$

This doesn't involve y, so we should try to graph the x and z part and then "draw the walls" by including all possible y-values.

First, complete the square to obtain $(x-1)^2 + 4z^2 = 16$, or perhaps easier, $((x-1)/4)^2 + (z/z)^2 = 1$ This you should recognize: it's an ellipse. It looks like this:



Problem 4. *Consider the three planes given by the following equations:*

$$x + 2y + 3z = 0$$
$$x - y + z = 2$$
$$x + 2y + 3z = 6.$$

a) *Two of these planes are parallel: which two? Describe the intersection of these planes.*

It's the first and the third: the way you can tell is that they both have the same normal vector $\mathbf{n} = \langle 1, 2, 3 \rangle$.

b) *The first and second planes intersect in a line. Give a parametrization of this line, and check that your line is actually contained in the first plane.*

First we need to find a point on the line. There are many points, and you can solve for one however you want. One way is just plug in $z = 0$ and then figure out what x and y have to be. You can also just guess one, which is what I am about to do: $(3, 0, -1)$.

Now we need to find the direction vector. How to do that? Well, it's perpendicular to both of the normal vectors, which are $\langle 1, 2, 3 \rangle$ and $\langle 1, -1, 1 \rangle$. So we can use the cross product $\mathbf{v} = \langle 1, 2, 3 \rangle \times \langle 1, -1, 1 \rangle$ as our normal vector! This guy is $\mathbf{v} = \langle 5, 2, -3 \rangle$.

So our line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 3, 0, -1 \rangle + t \langle 5, 2, -3 \rangle = \langle 3 + 5t, 2t, -1 - 3t \rangle .$$

To see whether it indeed line in the plane, just plug this in to the equation for the plane. You get $x + 2y + 3z = (3 + 5t) + 2(2t) + 3(-1 - 3t) = 0$, so it works as it should. You could check the same thing with the second plane if you were so inclined.