Math 210 (Lesieutre)
12.3: Limits and continuity
February 6, 2016

**Problem 1.** a) *Evaluate the limit:*

$$\lim_{(x,y)\to(1,2)} 2x^2 + \sqrt{xy}$$

This one we can do using the rules for limits.

$$\lim_{(x,y)\to(1,2)} 2x^2 + \sqrt{xy} = \lim_{(x,y)\to(1,2)} 2x^2 + \lim_{(x,y)\to(1,2)} \sqrt{xy}$$
$$= 2\left(\lim_{(x,y)\to(1,2)} x^2\right)^2 + \sqrt{\lim_{(x,y)\to(1,2)} xy} = 2 + \sqrt{2}.$$

b) *Evaluate the limit:*

$$\lim_{(x,y)\to(3,0)} \frac{3x^2 + y}{x + y}$$

This is another one where you can just "plug in" and things will work OK. The important thing to remember is that to take a limit of a quotient, you can just take quotient of the limits, unless the limit of the denominator is 0!

Since we're looking at the point $(3, 0)$, the limit is just 9 (what you get when you plug in $x = 3$ and $y = 0$).

**Problem 2.** *Evaluate the following limits along the path $y = mx$. Does the limit exist?*

a)

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2}$$

This gives us

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} \frac{x^4 + y^4}{x^2 + y^2} = \lim_{x\to 0} \frac{x^4 + (mx)^4}{x^2 + (mx)^2}$$
$$= \lim_{x\to 0} \frac{(1+m^4)x^4}{(1+m^2)x^2} = \lim_{x\to 0} \frac{1+m^4}{1+m^2}x^2 = 0,$$

no matter what $m$ is. So at least for these paths, the limit doesn't depend on the path. We can't really be sure from this that the limit actually exists, but in fact it does.

b)

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2 + y^2}$$

Let's again use $y = mx$ as our path.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x+mx)^2}{x^2+(mx)^2}$$
$$= \lim_{x\to 0} \frac{(1+m)^2}{1+m^2} = \frac{(1+m)^2}{1+m^2}.$$

This answer depends on $m$, which means that the limit doesn't exist.

**Problem 3.** a) *Give an example of a function $f(x,y)$ that is* not *continuous at $(1,2)$. Can you come up with an example that is continuous for all $(x,y)$ other than $(1,2)$?*

One function you could use is

$$f(x,y) = \frac{1}{(x-1)^2+(y-2)^2}.$$

As $(x,y)$ gets closer and closer to $(1,2)$, this gets closer and closer to infinity, and the function is not defined there.

b) *Give an example of a function $g(x,y)$ that is* not *continuous for any $(x,y)$ satisfying $y = x^2$.*

You can use a similar idea here as on the last one: think about the function

$$g(x,y) = \frac{1}{y-x^2}.$$

If $y = x^2$, you can't plug it in to this equation.

**Problem 4.** *Is the function*

$$p(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

*continuous at $(0,0)$?*

Being continuous means three things: the function is defined at $(0,0)$, the limit exists, and the limit is equal to the function. It is defined there ($p(0,0) = 0$ by fiat). So we need to think about the limit. Let's try the limit along the line $y = mx$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to 0} \frac{x^2(mx)^2}{x^4+(mx)^4}$$
$$= \lim_{x\to 0} \frac{(m^2)}{(1+m^4)} = \frac{m^2}{1+m^4},$$

so the direction limit depends on the path, which means that the limit doesn't exist, and the function isn't continuous.

**Problem 5.** *Set up the integral for the arc length of* $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ *for* $0 \le t \le \pi$.

We have

$$\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$$
$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1,$$

and so

$$L = \int_0^\pi \sqrt{|\mathbf{r}'(t)|}\, dt = \int_0^\pi 1\, dt = \pi.$$