Math 210 (Lesieutre)
12.4/12.5: Partial derivatives and the chain rule
February 8, 2017

Problem 1. *Compute the indicated partial derivatives:*

a) $\frac{\partial}{\partial x}(x^2y)$
It's $2xy$: just think of this as being a constant times x^2, and the derivative is the constant (i.e. y) times $2x$.

b) $\frac{\partial}{\partial y}(x^2y)$
This one is x^2, just as the derivative of $7y$ is 7 – you have to get used to thinking of the x's as constants.

c) f_y, *where* $f(x,y,z) = e^{xyz}$
$(xz)e^{xyz}$: treat the x and z in the exponent as a constant, and use the chain rule.

d) g_y, *where* $g(x,y) = y\cos x$
It's just $\cos x$.

Problem 2. *Consider the function* $f(x,y) = x\cos(xy)$. *Compute the four second partial derivatives.*

The first partials are:

$$\begin{aligned} f_x &= \cos(xy) - xy\sin(xy) \\ f_y &= -x^2\sin(xy) \end{aligned}$$

That gives the second partials as:

$$\begin{aligned} f_{xx} &= -y\sin(xy) - y(xy\cos(xy) + \sin(xy)) \\ &= -xy^2\cos(xy) - 2y\sin(xy) \\ f_{xy} &= -x\sin(xy) - x(xy\cos(xy) + \sin(xy)) \\ &= -x^2y\cos(xy) - 2x\sin(xy) \\ f_{yx} &= -(x^2(y\cos(xy)) + 2x\sin(xy)) \\ &= -x^2y\cos(xy) - 2x\sin(xy) \\ f_{yy} &= -x^3\cos(xy) \end{aligned}$$

The thing I want you to notice here is that $f_{xy} = f_{yx}$. This is always happens, and it's called "Clairaut's theorem".

Problem 3. *Suppose that* $f(x,y) = x^2 + y^2$, *and that* $x(t) = 2t$, $y(t) = \sin t$. *What is* $\frac{df}{dt}$*?*

Here we have to use the chain rule. The chain rule for one independent variable tells us that

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2x)(2) + (2y)(\cos t) = (2 \cdot 2t)(2) + (2\sin t)(\cos t) \\ &= 8t + 4\sin t\cos t = 8t + 2\sin(2t).\end{aligned}$$

Problem 4. *Suppose that* $f(x,y) = xy^2$, *and* $x(s,t) = 2s + t$ *and* $y(s,t) = s\cos t$. *Compute the partial derivative* $\frac{\partial f}{\partial s}$.

This is a little more painful. Again we have to use the chain rule.

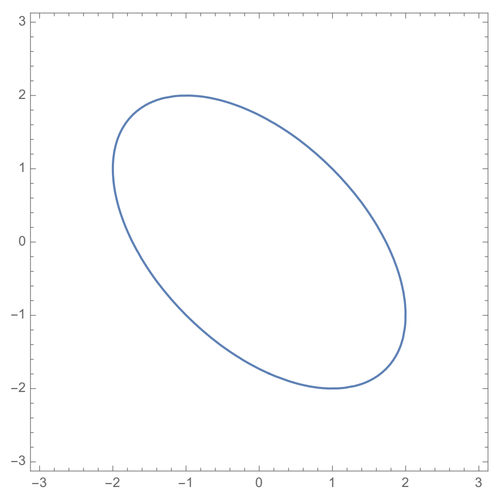
$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= (y^2)(2) + (2xy)(\cos t) = 2(s\cos t)^2 + (2)(2s+t)(s\cos t)(\cos t) \\ &= 2s^2\cos^2 t + (4s^2 + 2st)\cos^2 t = 6s^2\cos^2 t + 2st\cos^2 t.\end{aligned}$$

Problem 5. *Consider the ellipse defined by* $F(x,y) = 0$, *where* $F(x,y) = x^2 + xy + y^2 - 1$. *Compute* $\frac{dy}{dx}$.

The formula for implicit differentiation gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{2y+x}.$$

Let's make sure we actually understand what this is saying. Here's the ellipse:



(It's tilted because of the xy-term; you probably haven't plotted one like that, and it's tough until you've taken Math 310.)

One point on the ellipse is $(x, y) = (1, 1)$. Our formula is telling us that $\frac{dy}{dx}$, which is the slope of the tanegnt line, is given by $-\frac{2x+y}{2y+x} = -\frac{3}{3} = -1$ at this point. That appears to match up with the figure. So we were able to find the slope of the tangent line, even though we don't actually have a formula for y as a function of x.