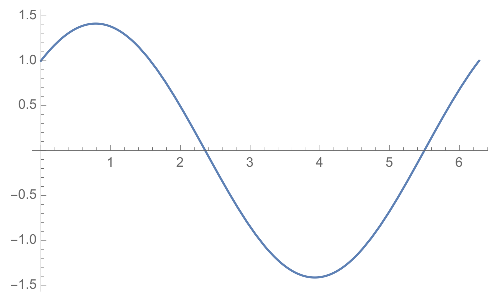
Math 210 (Lesieutre)
12.6 Directional derivatives, and some review
February 13, 2017

Problem 1. *Suppose that a function has gradient* $\nabla f(0,0) = (1,1)$.

a) *What is the directional derivative of this function in a direction with angle* θ*?*
The unit vector we want is $\langle \cos\theta, \sin\theta \rangle$, and so the directional derivative is

$$\nabla f(0,0) \cdot \mathbf{u} = \cos\theta + \sin\theta.$$

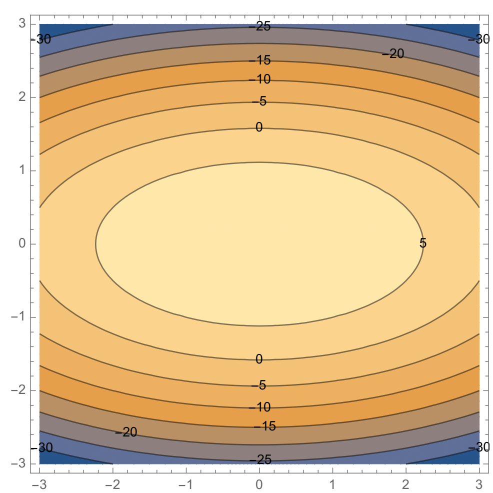
b) *Plot the directional derivative for* $0 \le \theta \le 2\pi$*. For what* θ *is it maximized? Zero?*
Here's a plot:



It's maximized at $\theta = \pi/4$, which is the direction parallel to ∇f. It's 0 at $3\pi/4$ and $7\pi/4$, which is orthogonal to the gradient.

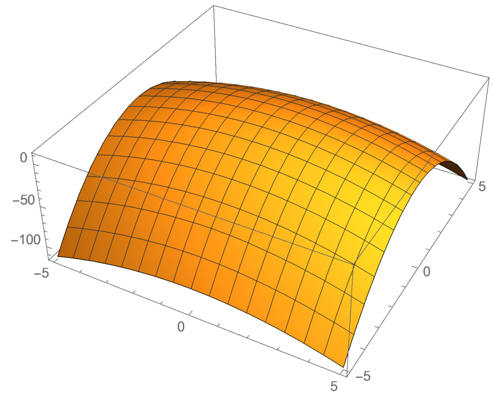
Problem 2. *Consider the function* $f(x,y) = 10 - x^2 - 4y^2$.

a) *Sketch some level curves of* $f(x,y)$*, including the level curve with* $z = 5$*.*
The level curves look like $x^2 + 4y^2 = C$, which are ellipses. Here are a few.



$z = 5$ is the innermost one that's marked. Notice that the point $(1, 1)$ is contained in this level curve.

Here's a graph of the whole surface, for what it's worth:



b) *Find the unit vectors in the directions of steepest ascent and descent at the point* $(1, 1)$. *Do your answers make sense?*

The gradient is given by $\langle -2, -8 \rangle$, which means steepest ascent is in the direction of $\langle -2, -8 \rangle$. A unit vector in this direction is $\left\langle -2/\sqrt{68}, -8/\sqrt{68} \right\rangle$ (a mess, sorry).

Looking at this on the plot, this vector points "inwards" on the ellipse. Makes sense: the whole surface is shaped like a "hill", and the inwards direction is uphill.

Steepest descent is in the opposite direction, which is $\left\langle 2/\sqrt{68}, 8/\sqrt{68} \right\rangle$. This is "downhill", the direction a ball would roll if placed on the graph.

c) *Find the directional derivative in the direction of steepest ascent. Is this steeper than the answers you got for directional derivatives earlier in the problem?*

The directional derivative is given by

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u} = \langle -2, -8 \rangle \cdot \left\langle -2/\sqrt{68}, -8/\sqrt{68} \right\rangle = 68/\sqrt{68} = \sqrt{68}.$$

This is slightly more than the 8 that we got in part (c), so it's plausible that this is indeed the direction of steepest ascent.

d) *Find a direction that is tangent to the level curve* $z = 5$ *at the point* $(1, 1)$. *What is the directional derivative in this direction?*

We want a direction that's perpendicular to the gradient. One option is $\left\langle 8/\sqrt{68}, -2/\sqrt{68} \right\rangle$, which will work because the dot product is 0. This points to the right and slightly down, which looks plausible for a tangent direction to the level curve at $(1, 1)$, based on the picture.

The directional derivative is 0, because it's tangent to a level curve: the function doesn't change in this direction. Taking the dot product with the gradient confirms this.

Problem 3. *Find the parametrization for a line which...*

a) *Has* $\mathbf{r}(0) = (1, 2, 3)$ *and* $\mathbf{r}(1) = (1, -2, 1)$.
For the point $\mathbf{r}_0$, use $1, 2, 3)$. For the direction, use $(1, -2, 1) - (1, 2, 3) = (0, -4, -2)$. So the equation is

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t = \langle 1, 2, 3 \rangle + t \langle 0, -4, 2 \rangle = \langle 1, 2 - 4t, 3 + 2t \rangle .$$

b) *Normal to the plane* $3x - 2y + z = 0$ *and passes through the origin.*
We want the line to be in the direction of the normal vector, so $\mathbf{v} = \langle 3, -2, 1 \rangle$.
The point is $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$, and so:

$$\mathbf{r}(t) = \langle 0, 0, 0 \rangle + \langle 3, -2, 1 \rangle t = \langle 3t, -2t, t \rangle .$$

c) *The intersection of the planes* $x + y + z = 3$ *and* $x - y + 2z = 1$.
This time the direction should be the cross product of the normal vectors, which is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle$. This comes out to

$$\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -1, -2 \rangle .$$

We also need to find a point on the line. There are many, so to narrow down our search and make it so there's only one answer (which is then easy to find), let's try to find a point with $z = 0$. Then we need $x + y = 3$ and $x - y = 1$. The solution is $x = 2$, $y = 1$, and so our point is $\langle 2, 1, 0 \rangle$. That means the line in question is given by

$$\mathbf{r}(t) = \langle 2, 4, 8 \rangle + \langle 1, 4, 12 \rangle t = \langle 2 + t, 4 + 4t, 8 + 12t \rangle .$$

d) *Tangent to the curve* $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ *at* $t = 2$.
The direction is given by the derivative, which is $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. We want to plug in $t = 2$ to get the direction vector at that time, and so the direction of the tangent line is $\mathbf{v} = \langle 1, 4, 12 \rangle$.
What point does the tangent line need to go through? It's $\mathbf{r}(2)$ itself, which is $\langle 2, 4, 8 \rangle$. So our equation for the line is

$$\ell(t) = \langle 2, 4, 8 \rangle + \langle 1, 4, 12 \rangle t = \langle 2 + t, 4 + 4t, 8 + 12t \rangle .$$

(Note: I'm calling the line $\ell(t)$ since $\mathbf{r}(t)$ was already in use for the original curve. It doesn't matter what you call it, as long as you're clear about what you're doing.)

Problem 4. *Consider the two vectors* $\mathbf{u} = \langle 1, 2, 3 \rangle$ *and* $\mathbf{v} = \langle -1, -1, -1 \rangle$.

a) *What is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*?*
We know $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \, |\mathbf{v}| \cos\theta$, and so the angle is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \, |\mathbf{v}|} \right) = \cos^{-1}\left(\frac{-6}{\sqrt{14}\sqrt{3}} \right) .$$

b) *If a triangle has* $(0,0,0)$ *as a vertex, with* $\mathbf{u}$ *and* $\mathbf{v}$ *the two edges from this vertex, what is the vector for the third edge?*

It's $\mathbf{u}-\mathbf{v} = \langle 2,3,4\rangle$ (or the other way, depending on which direction we want the edge vector to point).

c) *What is the area of this triangle?*

Remember that magnitude of the cross product gives the area of the parallelogram spanned by the vectors, and we want half of that.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & -1 & -1 \end{vmatrix} = \langle 1,-2,1\rangle .$$

So

$$\text{area} = \frac{1}{2}\left|\langle 1,-2,1\rangle\right| = \frac{1}{2}\sqrt{6}.$$