Math 210 (Lesieutre)
12.8: Maxima and minima
February 20, 2017

Problem 1. *A cylinder has radius* 2 *and height* 3.

a) *Suppose that the radius and height each increase by* 0.1. *Approximately how much will the volume change?*

The volume is given by $V(r,h) = \pi r^2 h$. We have $dV = V_r(a,b)\,dr + V_h(a,b)\,dh$. In this case $V_r = 2\pi r h$ so $V_r(2,3) = 12\pi$ and $V_h = \pi r^2$ so $V_h(2,3) = 4\pi$. Then

$$dV = V_r(a,b)\,dr + V_h(a,b)\,dh = (12\pi)(0.1) + (4\pi)(0.1) = 1.6\pi.$$

Let's check it. In fact the cylinder has volume 12π, and the new cylinder has volume 13.671π. The increase is 1.671π, which is just about what we expected. (This "true increase" is what the book calls Δz.)

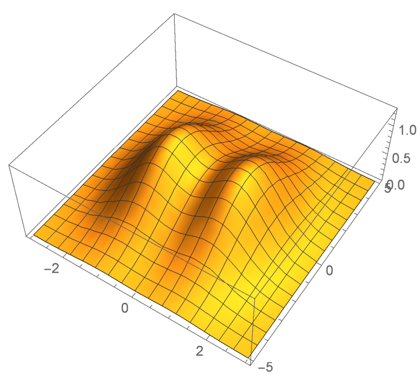
b) *Give a formula for the linear approximation to* $V(r,h)$ *near* $(r,h) = (2,3)$.

This one's supposed to be a review of last time. The formula for the linear approximation is

$$L(r,h) = V(2,3) + V_r(2,3)(r-2) + V_h(2,3)(h-3) = (12\pi)(r-2) + 4\pi(h-3).$$

Problem 2. *Here is a graph of the function.*

$$f(x,y) = e^{-(x-1)^2-(y/3)^2} + e^{-(x+1)^2-(y/3)^2}.$$



How many critical points can you identify on the graph? Are they maxima, minima, or saddle points?

I see three: there are two maxima, which are the two "mountain peaks", and there is a saddle point, which is the point directly between them. (One way to think about this: a critical point is a point where you could balance a marble on the graph and it wouldn't roll in any direction.) There are no local minima.

Problem 3. *For each of the following functions, compute all four second derivatives. Check that each function has a critical point at $(0,0)$, and classify it as a maximum, a minimum, or a saddle point.*

Let me preface these solutions with a warning. I made all the functions in this example very simple (just quadratics), which has the side effect that the second derivatives are all constant. This is not how things work in general! For other functions, you will need to plug in the (a,b) for your critical points into $f_{xx}(a,b)$ etc. before using the test.

a) $f(x,y) = x^2 + y^2$
Here we get

$$\begin{aligned} f_x &= 2x \\ f_y &= 2y \\ f_{xx} &= 2 \\ f_{xy} &= f_{yx} = 0 \\ f_{yy} &= 2 \end{aligned}$$

Plugging in $(x,y) = 0$, we get $f_x(0,0) = f_y(0,0) = 0$, as needed, so it is a critical point, as claimed by the problem statement.
To figure out whether it's a max or a min, we need to use the 2d version of the second derivative test. We have

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4,$$

which is positive. So it's either a maximum or a minimum, and since $f_{xx} > 0$, it's a minimum.

b) $f(x,y) = x^2 - y^2$
This time,

$$\begin{aligned} f_x &= 2x \\ f_y &= -2y \\ f_{xx} &= 2 \\ f_{xy} &= f_{yx} = 0 \\ f_{yy} &= -2 \end{aligned}$$

As before, plugging in $(x,y) = 0$, we get $f_x(0,0) = f_y(0,0) = 0$, so this is also a critical point.
Now we compute the determinant of the second partials. This time,

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4.$$

This is negative, so it's a saddle point. Roughly, what's going on is: if you fix $y = 0$ and just think of it as a function of x, it's a minimum (the function is just x^2, after all). If you fix $x = 0$ and think of it as a function of y, it's a maximum (since the function is $-y^2$). So whether the function increases or decreases depends on which direction you move.

c) $f(x,y) = -x^2 - 2y^2$
This time,

$$\begin{aligned} f_x &= -2x \\ f_y &= -4y \\ f_{xx} &= -2 \\ f_{xy} &= f_{yx} = 0 \\ f_{yy} &= -4 \end{aligned}$$

As before, plugging in $(x,y) = 0$, we get $f_x(0,0) = f_y(0,0) = 0$, so this is also a critical point.
Now we compute the determinant of the second partials. This time,

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} = 8.$$

positive, so it's either a max or a min. since $f_{xx} < 0$, in this case we know it has to be a maximum.

d) $f(x,y) = x^2 + xy + y^2$
This one's a little more interesting.

$$\begin{aligned} f_x &= 2x + y \\ f_y &= x + 2y \\ f_{xx} &= 2 \\ f_{xy} &= f_{yx} = 1 \\ f_{yy} &= 2 \end{aligned}$$

As before, plugging in $(x,y) = 0$, we get $f_x(0,0) = f_y(0,0) = 0$, so this is also a critical point.
Now we compute the determinant of the second partials. This time,

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

Positive, so max or min. To tell which, we check f_{xx}. Since it's positive, the point is a minimum.

Problem 4. *For each of the following functions, find the critical points. Pick one, and determine whether it is a maximum, minimum, or saddle point.*

a) $x^4 + y^4 - 16xy$
We have $f_x = 4x^3 - 16x$ and $f_y = 4y^3 - 16y$. To get both 0, we need $x^3 - 4x = 0$ and $y^3 - 4y = 0$, so $x = 0, \pm 2$ and $y = 0, \pm 2$. Any combination of these works, so the function has nine critical points.

The problem only asks us to look at one of these points, so let's use $(2, 2)$. We have

$$f_{xx} = 12x - 16$$
$$f_{xy} = 0$$
$$f_{yy} = 12y - 16.$$

At $(2, 2)$, this gives $f_{xx}(2, 2) = 8$ and $f_{yy}(2, 2) = 8$
$D(2, 2) = f_{xx}f_{yy} - f_{xy}^2 = 64$, which is positive. That means it's a max or a min. Since $f_{xx} > 0$, it's a minimum.

b) $f(x, y) = x^2 - 2x + y^2 - 4y + xy + 5$
We have

$$f_x = 2x - 2 + y,$$
$$f_y = 2y - 4 + x.$$

We need both of these to be 0, so $2x + y = 2$ and $x + 2y = 4$. Solving the linear system using whatever your favorite method is (probably just eliminating one of the variables), we get $x = 0$ and $y = 2$. This is the only critical point.
Is it a maximum or a minimum? Well,

$$f_{xx} = 2$$
$$f_{xy} = f_{yx} = 1$$
$$f_{yy} = 2$$

look at the determinant:

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

It's either or maximum or a minimum. To know which, we have $f_{xx} = 2$, so it's a minimum.

c) $g(x, y) = x^2ye^{-x^2-y^2}$.
This is going to be fun! Let's get all of the derivatives out of the way right off the bat.

$$g_x = x^2ye^{-x^2-y^2}(-2x) + 2xye^{-x^2-y^2} = 2(2xy - 2x^3y)e^{-x^2-y^2},$$
$$g_y = x^2e^{-x^2-y^2} + (x^2y)e^{-x^2-y^2}(-2y) = (x^2 - 2x^2y^2)e^{-x^2-y^2}.$$

So critical points happen whenever $g_x = g_y = 0$. Since $e^{-x^2-y^2}$ is never 0, we need to solve $2(2xy - 2x^3y) = 0$ and $x^2 - 2x^2y^2 = 0$. So $xy(1 - x^2) = 0$ and $x^2(1 - 2y^2) = 0$.
There are two possibilities: either $x = 0$ and y is another, or $y = \pm\frac{1}{\sqrt{2}}$ and $x = \pm 1$.
I'm not actually going to figure out what are the max and mins and saddle points. The point of this problem is simply to scare you: sometimes there are infinitely many critical points!