Math 210 (Lesieutre)
13.4: Triple integrals
March 3, 2017

Problem 1. *Compute the following triple integrals over rectangular regions.*

a) $\int_0^1 \int_1^2 \int_0^2 xyz\, dz\, dy\, dx$
Inner:

$$\int_0^2 xyz\, dz = xy\frac{z^2}{2}\Big|_0^2 = 2xy.$$

Middle:

$$\int_1^2 2xy\, dy = xy^2\Big|_{y=1}^2 = (4x) - (1x) = 3x.$$

Outer:

$$\int_0^1 3x\, dx = \frac{3x^2}{2}\Big|_0^1 = \frac{3}{2}.$$

That's our final answer.

b) $\int_0^2 \int_1^2 \int_0^1 xyz\, dx\, dy\, dz$
Inner:

$$\int_0^1 xyz\, dx = yz\frac{x^2}{2}\Big|_{x=0}^1 = \frac{yz}{2}.$$

Middle:

$$\int_1^2 \frac{yz}{2}\, dy = \frac{zy^2}{4}\Big|_{y=1}^2 = \frac{z2^2}{4} - \frac{z1^2}{4} = \frac{3z}{4}.$$

Outer:

$$\int_0^2 \frac{3z}{4}\, dz = \frac{3z^2}{8}\Big|_0^2 = \frac{3}{2}.$$

Of course, you're supposed to notice that this is the same integral as the one from part (a), but with the order changed. We got the same answer, which is reassuring. In fact, there are four other ways to rearrange it, and they'll all give you the same thing.

Problem 2. *Set up bounds for integrating a function $f(x, y, z)$ on a cylinder of height 3 and radius 2, with base centered at $(0, 0, 0)$.*

This one we want to use

$$\int_{z=0}^3 \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z)\, dx\, dy\, dz$$

(Note that sometimes people like to skip the "$x =$", "$y =$", "$z =$" at the bottom of the integral, since you can surmise which is which from the order of the d's. But I think this makes it easier to keep things straight.)

Problem 3. *Evaluate the triple integral*

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} yz \, dz \, dy \, dx$$

Inner:

$$\int_0^{2-x} yz \, dz = y \frac{z^2}{2}\Big|_{z=0}^{2-x} = y\frac{(2-x)^2}{2}.$$

Middle:

$$\int_0^{\sqrt{1-x^2}} y\frac{(2-x)^2}{2} \, dy = \frac{y^2}{2} \frac{(2-x)^2}{2}\Big|_{y=0}^{\sqrt{1-x^2}} = \frac{(1-x^2)(2-x)^2}{4}.$$

Outer:

$$\int_0^4 \frac{(1-x^2)(2-x)^2}{4} \, dx = -\frac{144}{5}.$$

The last integral is a hassle: I don't see any better way to do it than to just expand it the whole way.

Problem 4. *Change the bounds on the following integral from $dx\,dy\,dz$ to $dy\,dx\,dz$.*

$$\int_0^4 \int_0^1 \int_0^{2y} f(x,y,z) \, dx \, dy \, dz.$$

This is a triangular prism. The base is a triangle: y goes from 0 to 1, and x from 0 to $2y$, which gives the triangle with vertices at $(0,0)$, $(2,1)$, and $(0,1)$. In the other order, we get

$$\int_0^4 \int_0^2 \int_{x/2}^1 f(x,y,z) \, dy \, dx \, dz.$$