Math 210 (Lesieutre)
Change of variables in multiple integrals
March 27, 2017

Problem 1. *Consider the transformation T given by $x = 2u$, $y = 4u + v$.*

a) *Let S be the region in uv-plane given by the unit square. To what region R in the xy-plane does T send S?*

We actually did some of these last semester: this particular change of coordinates is a linear transformation, given by the matrix $\left(\begin{smallmatrix} 2 & 0 \\ 4 & 1 \end{smallmatrix}\right)$. In this case, perhaps the easiest way to see what it does is to think about what happens to each of the edges of the rectangle.

The bottom edge is $v = 0$, $0 \leq u \leq 1$, which turns into $x = 2u$, $y = 4u$, with $0 \leq u \leq 1$. This is a line from $(0, 0)$ to $(2, 4)$. Making a similar check for the other variables, we conclude that the region is given by a paralellogram with vertices at $(0, 0)$, $(0, 1)$, $(2, 4)$, and $(2, 5)$ in the xy-plane.

When you're dealing with a linear transformation, it will always turn the unit square into a parallelogram, and to figure out the vertices you can just check what the transformation does to the corners of the square. This is a pretty specific situation, but it's one that occurs frequently and might be worth remembering.

b) *What is the Jacobian for the transformation T?*

It's given by

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2.$$

c) *Compute $\iint_R \sqrt{2x(y - 2x)}\, dA$.*

Using the substitution rule,

$$\begin{aligned} \iint_R \sqrt{2x(y - 2x)}\, dA &= \iint_S \sqrt{4uv}\, |J(u, v)|\, dA \\ &= \int_0^1 \int_0^1 \sqrt{4uv}(2)\, du\, dv \\ &= \int_0^1 \frac{8}{3}\sqrt{v}\, dv = \frac{16}{9}. \end{aligned}$$

Problem 2. *Consider the transformation T given by $x = u \cos v$, $y = u \sin v$.*

a) *Let R be the top half of a circle of radius 2, in the xy-plane. What region in the uv-plane is mapped to R by the transformation T?*

This is easy: our u and v are just polar coordinates with different names! The region we want is a rectangle with $0 \leq u \leq 2$ (that's the radius) and $0 \leq v \leq \pi$ (that's the angle).

b) *What is the Jacobian for the transformation T?*
It's given by

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \cos^2 v + u \sin^2 v = u.$$

That means that when we integrate, the $|J(u,v)|\, du\, dv$ replacing $dx\, dy$ will be $u\, du\, dv$. This is the $r\, dr\, d\theta$ that we already know, but by another name!

c) *Compute* $\iint_R e^{-x^2-y^2}\, dA$
Using the substitution rule,

$$\begin{aligned} \iint_R e^{-x^2-y^2}\, dA &= \iint_S e^{-u^2}\, |J(u,v)|\, dA \\ &= \int_0^\pi \int_0^2 e^{-u^2} \cdot u\, du\, dv \\ &= \cdots \\ &= \frac{1}{2}\left(1 - e^{-4}\right)\pi. \end{aligned}$$

Here's the point of this problem: switching integrals into polar coordinates, like we've been doing for awhile, actually just boils down to being a special case of this business of making a change of variables and using the Jacobian.

Problem 3. *Make a substitution to evaluate the integral of* $\sqrt{\frac{x+y}{x-y}}$ *over a square* R *with vertices at* $(2,0)$*,* $(3,1)$*,* $(3,-1)$*, and* $(4,0)$*.*

First order of business is to guess our change of variables. The natural guess would seem to be $u = x + y$ and $v = x - y$; that will make the integrand simpler. Remember that we're supposed to give x and y in terms of u and v rather than the other way around, but we can get that from these equations. Adding together, we get $u + v = 2x$, so $x = \frac{u+v}{2}$. Similarly, $y = \frac{u-v}{2}$.

The hard part is to figure out the region in uv-plane that gets sent to R under this transformation. This is going to be the region that R is sent to under $u = x+y$ and $v = x-y$, so we might as well think about it that way. The corners of the square go to the (u,v) points $(2,2)$, $(4,2)$, $(2,4)$ and $(4,4)$. Since the transformation is linear, that's our region S: $2 \le u \le 4$ and $2 \le v \le 4$.

We also need to know the Jacobian. Using the formulas above, we get

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{1}{2}.$$

The integral is now

$$\iint_R \sqrt{\frac{x+y}{x-y}}\, dA = \int_2^4 \int_2^4 \sqrt{\frac{u}{v}} \frac{1}{2}\, dv\, du$$
$$= \int_2^4 (\sqrt{4} - \sqrt{2})\sqrt{u}\, du$$
$$= \frac{40}{3} - 8\sqrt{2}.$$

This integral could also be done without a change of variables, but it's quite messy. The set-up is

$$\int_2^3 \int_{-x+2}^{x-2} \sqrt{\frac{x+y}{x-y}}\, dy\, dx + \int_3^4 \int_{x-4}^{-x+4} \sqrt{\frac{x+y}{x-y}}\, dy\, dx.$$

(I split it into the left half of the square and the right half of the square.) The computer confirms that this gives the same result, but even it had to think about it for awhile.

Problem 4. *Let R be the region bounded by $x = -1$, $x = 1$, $y = x^2$, and $y = x^2 + 1$. Compute the area by integrating the function 1 over the region, using a substitution.*

It's nice when we have a region where the sides are constants, right? The bound $y = x^2 + 1$ could be thought of as $y - x^2 = 1$. So take $u = y - x^2$ and $v = x$. Again, we need to know x in terms of u and v to find the Jacobian, etc. So a little algebra is required: $x = v$ and $y = u + v^2$

Our region in the uv-plane is now just $0 \leq u \leq 1$ and $-1 \leq v \leq 1$. The annoying part is going to be the Jacobian, which is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 2v \end{vmatrix} = 2v.$$

Rewriting the integral:

$$\int_{-1}^1 \int_0^1 1\ |2v|\ du\, dv = 2.$$