Math 210 (Lesieutre)
14.3: Conservative vector fields, I
April 3, 2017

Problem 1. *Consider the two fields*

$$\mathbf{F}_1(x,y) = \left\langle x^2, y^2 \right\rangle, \quad \mathbf{F}_2(x,y) = \left\langle y^2, x^2 \right\rangle.$$

One of these is a conservative field, and the other one isn't. Which is which? Find the potential function.

Let's try to find a potential function. Our field is $f(x,y) = x^2$ and $g(x,y) = y^2$. If $\mathbf{F} = \nabla\phi$, then $\phi_x = x^2$. This means that $\phi = \frac{x^3}{3} + c(y)$ (here $c(y)$ could be any function, but it only depends on y, which makes its x partial 0). Then we want $\phi_y = y^2$. Since $\phi = \frac{x^3}{3} + c(y)$, we get $y^2 = c_y$, so that $c = \frac{y^3}{3}$. This gives our potential as $\phi(x,y) = \frac{x^3}{3} + \frac{y^3}{3}$.

How about the second one? Well, $f_x = y^2$, so $f(x,y) = xy^2 + g(y)$. This gives $f_y = 2xy + g_y$, which we want to be y^2. No function g_y is going to make this work, so this one isn't a gradient.

Problem 2. *Check whether the field* $\mathbf{F} = \langle \sin y, x\cos y + 1\rangle$ *is conservative. If it is, find a potential function.*

We have $f_y = \cos y$ and $g_x = \cos y$. Since there isn't anything funny going on (both functions are defined everywhere), this means that the field is going to be conservative. Now to actually find the potential function.

We want our function ϕ to have $\phi_x = \cos y$, so $\phi = x\cos y + c(y)$. Then $\phi_y = x\sin y + c_y(y)$. We want that to be $x\cos y + 1$, so it had better be the case that $c_y = 1$. That means that $c(y) = y$ (plus a constant, if you want). All told, this means that our fnuction is $\phi(x,y) = x\cos y + y$.

Problem 3. *Consider the vector field*

$$\mathbf{F}(x,y,z) = \langle yz, xz + ze^{yz}, xy + ye^{yz} + \sin z\rangle.$$

This field is conservative. Find a potential function.

We have $\phi_x = yz$, so $\phi = xyz + c(y,z)$.

We then have $\phi_y = xz + c_y(y,z)$, which should be $xz + ze^{yz}$. So $c_y = ze^{yz}$, yielding $c(y,z) = e^{yz} + d(z)$. This gives $\phi = xyz + e^{yz}$.

Now, $\phi_z = xy + ye^{yz} + d_z$, should should be $xy + ye^{yz} + \sin z$. So $d_z = \sin z$, and $d(z) = -\cos z$. Our potential function is therefore

$$\phi(x,y,z) = xyz + e^{yz} - \cos z.$$

Problem 4. *Consider the field* $\mathbf{F}_1 = \langle x^2, y^2\rangle$ *from the first problem. Compute the line integral* $\int_C \mathbf{F}_1 \cdot d\mathbf{r}$ *for two paths from* $(0,0)$ *to* $(1,1)$*: first, a straight line path. Second, a path that goes from* $(0,0)$ *to* $(1,0)$ *in a straight line, and then from* $(1,0)$ *to* $(1,1)$ *in a straight line.*

Let's do the first one first. We have $\mathbf{r}(t) = \langle t, t\rangle$ with $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 1, 1\rangle$. Plugging in $x(t) = t$ and $y(t) = t$, our integral becomes

$$\int_0^1 \left\langle t^2, t^2 \right\rangle \cdot \langle 1, 1\rangle \, dt = \int_0^1 2t^2 \, dt = \frac{2}{3}.$$

Now comes the second path. We have to integrate over the two parts separately, then add the result. For the first piece, $\mathbf{r}(t) = \langle t, 0\rangle$ with $0 \le t \le 1$. This gives us $\mathbf{r}'(t) = \langle 1, 0\rangle$, and we have to integrate

$$\int_0^1 \left\langle t^2, 0 \right\rangle \cdot \langle 1, 0\rangle \, dt = \int_0^1 t^2 \, dt = \frac{1}{3}.$$

For the second piece, $\mathbf{r}(t) = \langle 0, t\rangle$ and $\mathbf{r}'(t) = \langle 0, 1\rangle$. The corresponding integral is

$$\int_0^1 \left\langle 0, t^2 \right\rangle \cdot 0, 1 \, dt = \int_0^1 t^2 \, dt = \frac{1}{3}.$$

The total integral along this bent path is $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

We got the same answer for both paths. Is it a coincidence? No – this always happens for conservative fields. We'll talk about this next time.