Math 210 (Lesieutre)
Chapter 14 review
April 25, 2017

Problem 1. *Compute the flux of the vector field* $\mathbf{F} = \langle 2x^2, xz^4, \sin y\rangle$ *across a tetrahedron with vertices at the points* $(1,0,0)$, $(0,2,0)$, *and* $(0,0,3)$.

We'd have to do flux integrals over four different surfaces to get this, so of course it's a better idea to just use the divergence and compute the triple integral of the divergence over the tetrahedron. The divergence is easy to calculate for this one: it's

$$\nabla \cdot \mathbf{F} = 4x.$$

So we need to compute

$$\iiint_D 4x\, dV.$$

This is a trick review question: the first thing we need to do is come up with the equation for the plane! Call the points A, B, and C. To get the normal vector, we're going to use $AB \times AC$. We have $AB = \langle -1, 2, 0\rangle$ and $AC = \langle -1, 0, 3\rangle$. Then $AB \times AC = \langle -6, -3, -2\rangle$. A plane with this normal vector through $(1,0,0)$ has equation

$$-6(x-1) - 3(y-0) - 2(z-0) = 0.$$

This simplifies to $6x + 3y + 2z = 6$. Solving for z, we get $z = \frac{6-6x-3y}{2}$. The base in the xy-plane is obtained by taking $z = 0$, which gives $y = 2 - 2x$.

$$\int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{\frac{6-6x-3y}{2}} 4x\, dz\, dy\, dx.$$

This comes out to

$$\begin{aligned}
T &= \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{\frac{6-6x-3y}{2}} 4x\, dz\, dy\, dx \\
&= \int_{x=0}^{1} \int_{y=0}^{2-2x} \left(\frac{6-6x-3y}{2} 4x\right) dy\, dx \\
&= \int_{x=0}^{1} \int_{y=0}^{2-2x} 12x - 12x^2 - 6xy\, dy\, dx \\
&= \int_{x=0}^{1} 12x^3 - 24x^2 + 12x\, dx = 1.
\end{aligned}$$

Problem 2. *What does each of the following types of integral represent? Can you remember how to compute one?*

a) $\displaystyle\int_C f\,ds$

This is the integral of a scalar function along a path. To compute it, you need to parametrize the path by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then plug in x and y to the function f, and use $|\mathbf{r}'(t)|\,dt$ for ds. The bounds on the integral are the bounds on your parametrization, you're just integrating a function of t.

b) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$

This is a circulation integral, also known as a work integral, also known as the integral of a vector field along a path. The procedure is similar: parametrize it by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Plug in x and y to the vector field $\mathbf{F}$, to get a vector. Dot that vector with the vector $\mathbf{r}'(t)$, and integrate the result from $t = a$ to $t = b$.

c) $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds$

The procedure is mostly like the above, except you want to dot that vector with the vector $\mathbf{n}(t) = \langle y(t), -x(t)\rangle$, and integrate the result from $t = a$ to $t = b$.

d) $\displaystyle\iint_S f\,dS$

This is the integral over a scalar function over a surface. This is the one that's the most work of the whole bunch. You need to parametrize the surface S by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. Then compute $\mathbf{t}_u = \frac{\partial\mathbf{r}}{\partial u}$ and $\mathbf{t}_v = \frac{\partial\mathbf{r}}{\partial v}$, and use those to compute $|\mathbf{t}_u \times \mathbf{t}_v|$. At last, take $|\mathbf{t}_u \times \mathbf{t}_v|$, a scalar function involving a square root unless you're very fortunate.

To set up the double integral, use your bounds on the parametrization, then plug in the x, y, and z to $f(x, y, z)$ to get a function of u and v. For dS, you want to use $|\mathbf{t}_u \times \mathbf{t}_v|\,du\,dv$.

e) $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$

Start off like the above, compute up to $\mathbf{t}_u \times \mathbf{t}_v$. Plug in x, y, z to your field $\mathbf{F}$ to get a vector field in terms of u and v. Dot that with $\mathbf{t}_u \times \mathbf{t}_v$ to get a function of u and v. Integrate that over the bounds of your parametrization.

Problem 3. *Try to recall the main integration theorems from this chapter.*

a) *Fundamental theorem for line integrals.*

This one says that if $\mathbf{F}$ is a conservative field, so that $\mathbf{F} = \nabla\phi$ for some scalar function ϕ, and if C is a curve, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A),$$

where B is the endpoint of C and A is the beginning point of C.

b) *Green's theorem, circulation form.*

This time we have a region R and a curve C going around R, counterclockwise. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dA.$$

Here $\operatorname{curl} \mathbf{F}$ is the curl (duh). If we have a 2D field $\mathbf{F} = \langle f(x,y), g(x,y) \rangle$, then $\operatorname{div} \mathbf{F} = g_x - f_y$.

c) *Green's theorem, flux form.*

This is similar to the previous one. It says that

$$\oint_C \mathbf{F} \cdot \mathbf{n} = ds = \iint_R \operatorname{div} \, dA.$$

Here $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$ is the divergence. If we have a 2D field $\mathbf{F} = \langle f(x,y), g(x,y) \rangle$, then $\operatorname{div} \mathbf{F} = f_x + g_y$.

d) *Stokes' theorem*

This is a 3D thing. We have a surface S in 3D, and the boundary of the surface is a curve C (oriented appropriately – see the previous sheet).

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$

e) *Divergence theorem*

The divergence theorem related a triple integral to a double integral. Let D be a 3D region, with boundary a surface S. Then

$$\iiint_D (\nabla \cdot \mathbf{F}) \, dV = \oiint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

Problem 4. *(We'll vote on an integral to compute.)*