Math 310, Lesieutre
Midterm #1
September 30, 2015

Name: ________________________________

1. *Consider the following system of linear equations:*

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & & & - & 4x_4 & = & 1 \\ 2x_1 & + & 6x_2 & + & x_3 & - & 6x_4 & = & 2 \\ & & & & 2x_3 & + & 4x_4 & = & 0 \end{array}$$

(a) *(15 pts) Find the general solution to this system.*

Write down the augmented matrix for the system and do row reduction.

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 1 \\ 2 & 6 & 1 & -6 & 2 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow{R2-=2R1} \left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow{R3-=2R2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is echelon form, and we can read off the solution. The variables $x_1$ and $x_3$ are pivots. The general solution is

$$\begin{cases} x_1 = 1 - 3x_2 + 4x_4, \\ x_2 \text{ is free,} \\ x_3 = -2x_4, \\ x_4 \text{ is free.} \end{cases}$$

(b) *(10 pts) Express the general solution to the system in parametric vector form.*

Let $s$ and $t$ be parameters for the free variables $x_2$ and $x_4$. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 3s + 4t \\ s \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

This is parametric vector form.

2. *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by*

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) *(5 pts) What is*

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)?$$

*Hint:* $\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\2\\0\end{smallmatrix}\right] - 2\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]$.
It's a linear map, so

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\\0\end{bmatrix}\right) - 2T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$
$$= \begin{bmatrix}4\\-4\end{bmatrix} - 2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}.$$

(b) *(10 pts) Write down the matrix for the linear transformation $T$. If you aren't sure about your answer to (a), you can assume* $T\left(\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}3\\-3\end{smallmatrix}\right]$.
It's given by

$$A = \left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_1)\right] = \begin{bmatrix}2 & 1 & 1\\-2 & -1 & -1\end{bmatrix}.$$

(c) *(5 pts) Is this linear transformation onto? Justify. Explain what your answer means about the transformation $T$.*
To check if it's onto, we need to find rref of $A$ and see whether there's a pivot in every row. Row reduction goes

$$\begin{bmatrix}2 & 1 & 1\\-2 & -1 & -1\end{bmatrix} \to \begin{bmatrix}2 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

There's not a pivot in the second row, which means that the map is not onto.

3. *Consider the vectors*

$$\mathbf{v}_1 = \begin{bmatrix}1\\-2\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\1\\1\end{bmatrix}, \quad \mathbf{v}_3 \begin{bmatrix}6\\-2\\h\end{bmatrix}$$

(a) *(10 pts) For what value(s) of $h$ are the three vectors* linearly dependent*?*
We can check this by row reducing the whole thing in terms of $A$. I'll include the augmented column of 0s.

$$\left[\begin{array}{ccc|c}1 & 2 & 6 & 0\\-2 & 1 & -2 & 0\\0 & 1 & h & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 2 & 6 & 0\\0 & 5 & 10 & 0\\0 & 1 & h & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 2 & 6 & 0\\0 & 1 & h & 0\\0 & 5 & 10 & 0\end{array}\right]$$
$$\to \left[\begin{array}{ccc|c}1 & 2 & 6 & 0\\0 & 1 & h & 0\\0 & 0 & 10-5h & 0\end{array}\right].$$

These vectors are linearly dependent if there's no pivot in the third column, which will happen when $10 - 5h = 0$, so $h = 2$.

(b) *(5 pts) For what value(s) of $h$ is the following matrix* invertible*?*

$$\begin{bmatrix} 1 & 2 & 6 \\ -2 & 1 & -2 \\ 0 & 1 & h \end{bmatrix}$$

*Justify your answer. (Hint: the columns of the matrix are the same as the vectors in (a))*

A matrix is invertible if its columns are linearly independent, which happens for $h \neq 2$ according to our answer to part (a).

4. (a) *(10 pts) Set up a system of linear equations that you could use to balance the following chemical reaction, and write down the augmented matrix. You do not need to solve it.*

$$Cl_2O + H_3N \longrightarrow NH_4Cl + N_2 + H_20$$

Give names to the coefficients in front of the various species:

$$x_1\mathrm{Cl_2O} + x_2\mathrm{H_3N} \longrightarrow x_3\mathrm{NH_4Cl} + x_4\mathrm{N_2} + x_5\mathrm{H_20}$$

Each element gives us an equation:

$$\begin{aligned} \text{Cl:} &\quad 2x_1 = x_3 \\ \text{O:} &\quad 2x_1 = x_5 \\ \text{H:} &\quad 3x_2 = 4x_3 + 2x_5 \\ \text{N:} &\quad x_2 = x_3 + 2x_4 \end{aligned}$$

The augmented matrix for the system is

$$\left[\begin{array}{ccccc|c} 2 & 0 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 3 & -4 & 0 & -2 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 \end{array}\right]$$

(b) *(10 pts) Suppose that 90% of people who are healthy one day are healthy the next day (while the rest become sick), and 60% of people who are sick one day are sick the next day (and the rest are healthy)*

*On day 0, there are 100 healthy people and 100 sick people. How many healthy and sick people are there on day 2? Compute the answer using a difference equation.*

We have $\mathbf{x}_{n+1} = A\mathbf{x}_n$, where

$$A = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}.$$

Then

$$\mathbf{x}_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$
$$\mathbf{x}_1 = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 130 \\ 70 \end{bmatrix}$$
$$\mathbf{x}_2 = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} 130 \\ 70 \end{bmatrix} = \begin{bmatrix} 145 \\ 55 \end{bmatrix}.$$

5. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}.$$

(a) *(10 pts) Compute the matrix $A^2$.*

We have

$$A^2 = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 32 & 57 \end{bmatrix}.$$

(b) *(10 pts) Compute the inverse of $A$ using row reduction. (You will only get half credit if you use the formula for inverse of a $2 \times 2$ matrix!)*

For this one, we want

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -7 & 2 \\ 0 & -1 & -4 & 1 \end{array}\right] \to$$
$$\left[\begin{array}{cc|cc} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right]$$

This shows that the inverse is

$$A^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix},$$

which is the same thing we'd get using the formula for the $2 \times 2$ case, because the determinant of this matrix is equal to $-1$.