

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

(a) [10] Compute an LU decomposition for A .

Row reduction goes

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}.$$

The row reduction steps we did were:

- Subtract 2 row 1 from row 2;
- Subtract 1 row 1 from row 3;
- Subtract -2 row 2 from row 3.

This gives

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}.$$

(b) [10] What is the determinant of A ?

You've already done row reduction on A , so use it! Just multiply together the pivots. The determinant is -3 .

2. Consider the matrix

$$B = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$$

(a) [12] Find the eigenvalues and eigenvectors of B .

The determinant is

$$\det(B - \lambda I) = (-1 - \lambda)(1 - \lambda) - 3 = \lambda^2 - 1 - 3 = \lambda^2 - 4.$$

The eigenvalues are 2 and -2

For $\lambda = 2$ we have

$$B - \lambda I = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix},$$

and the eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = -2$ it's

$$B - \lambda I = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix},$$

with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(You might've ended up with multiples of these eigenvectors instead; that works too.)

- (b) [8] Give a diagonalization of B (i.e. find P and D so $B = PDP^{-1}$, where D is diagonal).

We get P by sticking the eigenvectors as the columns of a matrix, and D by putting the eigenvalues as the diagonal entries:

$$P = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

(With the same caveat that you will have a different answer if you used multiples of these eigenvectors, or put them into the matrix in a different order.)

3. Row reduction on a matrix C yielded the echelon form

$$U = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- (a) [5] I'm not going to tell you the original matrix C . You still have enough information to find 2 out of the 3 spaces $\text{Row } C$, $\text{Col } C$, and $\text{Nul } C$. Which ones?

You can find $\text{Row } C$ and $\text{Nul } C$, because these are the same as $\text{Row } U$ and $\text{Nul } U$. To find $\text{Col } C$, you'd need to know the columns of the original matrix.

- (b) [10] For each of the spaces in your answer to (a), give a basis. What are the dimensions of the three subspaces? (Include the dimension of the one for which you did not find a basis.)

A basis for the row space is given by the nonzero rows of the echelon form U :

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

A basis for the nullspace is found using parametric vector form. There are two free variables, x_2 and x_4 . Parametrize them using s and t .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

A basis for the nullspace is the two vectors

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The nullspace has dimension 2, as does the row space: the basis we found for both of these had two vectors each. The column space has the same dimension as the row space (as it does for any matrix), and so is 2 as well.

- (c) [10] Let \mathcal{B} be the basis for Row C that you found in part (b). Another basis \mathcal{C} is given by the two vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find the change of basis matrix $\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{C}}$.

We want to use the formula $\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{C}} = [[\mathbf{c}_1]_{\mathcal{B}} \quad [\mathbf{c}_2]_{\mathcal{B}}]$. (The formula $\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{C}} = \mathcal{P}_{\mathcal{B}}^{-1} \mathcal{P}_{\mathcal{C}}$ isn't applicable here, since these aren't bases for \mathbb{R}^n : they're bases for a subspace of \mathbb{R}^n .)

It's easy to see what $[\mathbf{c}_1]_{\mathcal{B}}$ and $[\mathbf{c}_2]_{\mathcal{B}}$ are here:

$$\begin{aligned} \mathbf{c}_1 &= 1 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ \mathbf{c}_2 &= 0 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

So $[\mathbf{c}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $[\mathbf{c}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and we get

$$\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

4. (a) [10] Use cofactor expansion to compute the determinant of

$$E = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & -2 \\ 2 & 0 & 4 \end{bmatrix}.$$

(Half credit if you use some other method instead.)

Let's expand down the 2nd column:

$$\det E = -(1) \det \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix} + (-2) \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0 + 0 = 0.$$

(b) [5] *What does your answer to (a) tell you about the columns of E ?*

That the determinant is 0 means that the columns are linearly dependent.

5. (a) [10] *A Markov process is defined by the stochastic matrix*

$$M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}.$$

What is $\lim_{n \rightarrow \infty} M^n \mathbf{x}_0$ if $\mathbf{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$? (In other words, what vector does $M^n \mathbf{x}_0$ approach when n is very large?)

When n is large, this converges to the steady state of M . So we just need to find the steady state, by finding a vector in the nullspace of $M - I$.

$$M - I = \begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}.$$

A vector in the nullspace is $\begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}$. We want a state vector, so the entries need to sum to 1. To achieve that, we multiply our vector by 2 and obtain the final answer:

$$\mathbf{x} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}.$$

(b) [10] *Consider the three polynomials*

$$\mathbf{p}_1(t) = 1 + t, \quad \mathbf{p}_2(t) = 1 + 2t, \quad \mathbf{p}_3(t) = 2 + 3t.$$

Are these three polynomials linearly independent? Justify your answer. (Hint: think about the coordinate vectors for these polynomials in some basis.)

To check if they're independent, it's enough to check if the corresponding coordinate vectors are. A basis for the vector space of polynomials of degree at most 1 is given by 1 and t , and the coordinates for these polynomials with respect to that basis are:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

That's 3 vectors in \mathbb{R}^2 ; there's no way they're linearly independent. So the polynomials aren't either.