Bases as coordinate systems

Fact: if $\vec{v}_1, \dots, \vec{v}_d$ is a basis for a vector space / subspace, then any vector $\vec{x}$ can be written as a combination of $\vec{v}_1, \dots, \vec{v}_d$ in exactly one way:

$$\vec{x} = c_1\vec{v}_1 + \dots + c_d\vec{v}_d$$

Why? $\vec{v}_1, \dots, \vec{v}_d$ are a basis for the vector space V. that means that they span V, which means

$\vec{x} = c_1\vec{v}_1 + \dots + c_d\vec{v}_d$ for some values of c_i.

why is this unique? well, suppose

$\vec{x} = c_1\vec{v}_1 + \dots + c_d\vec{v}_d$

and $\vec{x} = e_1\vec{v}_1 + \dots + e_d\vec{v}_d$.

that means $\vec{0} = (c_1 - e_1)\vec{v}_1 + \dots + (c_d - e_d)\vec{v}_d$.

$\vec{v}_i$'s are a basis so linearly independent!

so it must be that $c_1 - e_1 = 0$, $c_2 - e_2 = 0, \dots$ $c_d - e_d = 0$

ie $c_1 = e_1, \dots, c_d = e_d$

If $\mathcal{B} = \vec{b}_1, \dots, \vec{b}_n$ is a basis for a vector space V

and $\vec{v}$ is a vector, then

$$\vec{v} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n$$ in exactly one way.

the weights $c_1, \dots, c_n$ are called the

$\mathcal{B}$-coordinates of $\vec{v}$.

write $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. this is the <u>coordinate vector</u> for $\vec{v}$.

Example: Vector space $\mathbb{R}^2$.

basis $\mathcal{B}$: $\underset{\vec{b}_1}{\underset{\uparrow}{\begin{pmatrix}3\\1\end{pmatrix}}}, \underset{\vec{b}_2}{\underset{\uparrow}{\begin{pmatrix}1\\2\end{pmatrix}}}$.

this is a basis ✓.

the vector $\vec{v} = \begin{pmatrix}11\\7\end{pmatrix}$ can be written as

$$\begin{pmatrix}11\\7\end{pmatrix} = 3\begin{pmatrix}3\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}.$$

so $\mathcal{B}$-coordinates of $\begin{pmatrix}11\\7\end{pmatrix}$ are $c_1 = 3$
$c_2 = 2$.

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}3\\2\end{bmatrix}$$

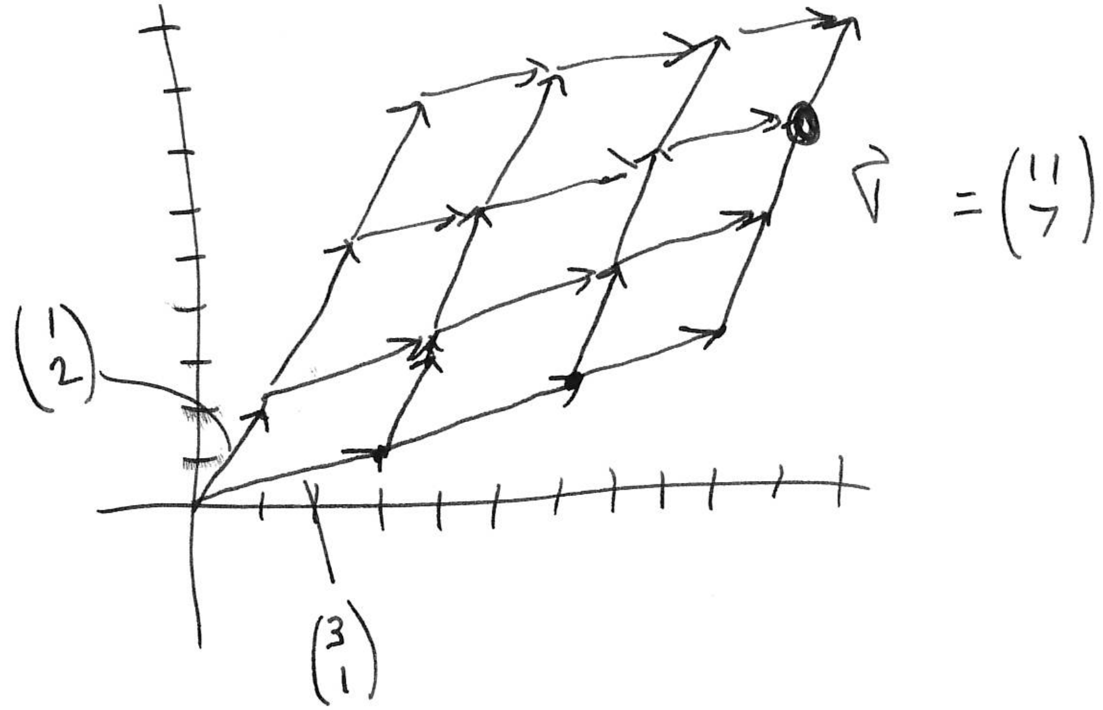
if $\vec{v} = \begin{bmatrix}2\\3/2\end{bmatrix}$, it's $\frac{1}{2}\begin{bmatrix}3\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix}$

$c_1 = 1/2$
$c_2 = 1/2$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}1/2\\1/2\end{bmatrix}$$

coordinates as "crooked graph paper"

for the basis $\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$



the vector with $\mathcal{B}$-coordinates $\begin{pmatrix}3\\2\end{pmatrix} = [\vec{v}]_{\mathcal{B}}$

is obtained by going 3 steps along $\vec{b}_1$,

2 steps along $\vec{b}_2$.

We've also talked about bases for other vector spaces.

e.g. $V = \mathbb{P}_2$ = polynomials of degree ≤ 2

$$at^2 + bt + c.$$

a basis for this vector space is

$\mathcal{B} = t^2, t, 1$

if we look at $\vec{p} = 3t^2 - 7t + 5$,

it's a combination:

$$(3)t^2 + (-7)t + (5).$$

the $\mathcal{B}$-coordinates of $\vec{p}$ are $\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}$.

$$[\vec{p}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}.$$

$(0)t^2 + (3)t + (5)$

$\vec{p} = 3t + 5$ $\quad [\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

this is cool! we can check if polynomials are lin independent by checking if coordinate vectors are.