Announcements

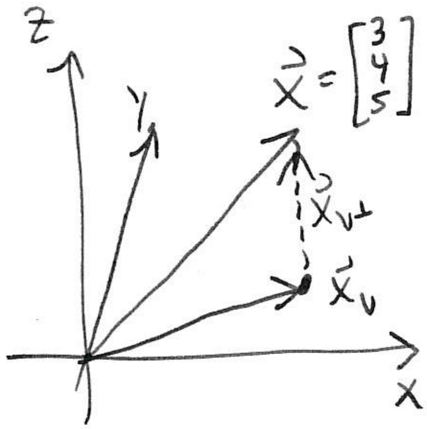
- Pick up your exam on the way out
- OH tomorrow cancelled; email to set a time. (Fri or Mon)
- Today: orthogonal projection.

Geometric observation:

$\vec{x}$ a vector in $\mathbb{R}^3$

$V =$ xy-plane, a subspace.

you can always write $\vec{x}$ as $\vec{x}_V + \vec{x}_{V^\perp}$, where $\vec{x}_V$ is in xy-plane, and $\vec{x}_{V^\perp}$ is perpendicular to V.



$$\vec{x}_V = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

$$\vec{x}_{V^\perp} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

the same thing can be done for any plane V, doesn't have to be xy-plane.

$$\underset{\text{in } V}{\vec{x}_V}$$

$$\vec{x} = \underset{\substack{\uparrow \\ \text{in } V}}{\vec{x}_V} + \underset{\substack{\uparrow \\ \text{in } V^\perp}}{\vec{x}_{V^\perp}}.$$

Finding $\vec{x}_V$, $\vec{x}_{V^\perp}$ is easy if V is xy-plane,

but there's a way to do it for any other plane.

If $W \subset \mathbb{R}^n$ is a subspace, and $\vec{w}_1, \dots, \vec{w}_p$

is an orthogonal basis for W, then any vector

$\vec{x}$ can be written as $\vec{x}_W + \vec{x}_{W^\perp}$, where $\vec{x}_W$ in W,

$\vec{x}_{W^\perp}$ perp to W.

$$\vec{x}_W = \underbrace{\frac{\vec{x} \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}}_{\text{scalar}} \vec{w}_1 + \underbrace{\frac{\vec{x} \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}}_{\text{scalar}} \vec{w}_2 + \dots + \underbrace{\frac{\vec{x} \cdot \vec{w}_p}{\vec{w}_p \cdot \vec{w}_p}}_{\text{scalar}} \vec{w}_p .$$

then $\vec{x}_{W^\perp} = \vec{x} - \vec{x}_W$.

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x_W is definitely in W, since it's a linear combo of

basis vectors.

Example

$W = xy\text{-plane}.$

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Formula says:

$$\vec{x}_W = \frac{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{3}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{4}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \quad \checkmark$$

$$\vec{x}_{W^\perp} = \vec{x} - \vec{x}_W = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}.$$

Notes:

1. $\vec{x}_W$ is also called $\text{proj}_W \vec{x}$.

2. $\vec{x}_W$ is the vector in W that's as close as possible to $\vec{x}$.
   (distance as small as possible)

3. - the formula required an orthogonal basis!
   - our methods for finding bases don't give orthogonal bases, most of the time.
   - Fri: Gram-Schmidt ~~elimination~~ orthonormalization: given a basis, convert to an orthogonal basis. this can be used in the formula.