Math 310, Lesieutre
Final exam practice
November 25, 2015

Name: ______________________________

This is a version of an old exam, with a couple questions replaced. Note: many topics are still not represented!

1. (a) Solve the following system of linear equations, putting your answer in parametric vector form:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 12 \\ -2x_1 & - & 2x_2 & + & x_3 & & & = & -1 \\ x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 10 \end{array}$$

   (b) Consider the matrix $A = \begin{bmatrix} 11 & 6 & 17 & 28 \\ -1 & -1 & -2 & -3 \\ 3 & 2 & 5 & 8 \end{bmatrix}$ with reduced echelon form $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   Given that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 186 \\ -21 \\ 54 \end{bmatrix},$$

   find the parametric vector form the the solutions to $A\mathbf{x} = \begin{bmatrix} 186 \\ -21 \\ 54 \end{bmatrix}$.

2. Suppose that $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 19 & 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 1 \end{bmatrix}$.

   (a) What is the determinant of A?

   (b) If $A\mathbf{x} = \mathbf{b}$, use Cramer's rule to find x_2.

3. Suppose that $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Compute $B = A^T A$ and find B^{-1}.

   (b) Explain why $A\mathbf{x} = \mathbf{b}$ is inconsistent, and write down the normal equations of the system.

   (c) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$.

   (d) What is the projection of $\mathbf{b}$ onto $\operatorname{Col} A$?

4. (a) Show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ are both bases for $\mathbb{R}^2$.

(b) Write down the change of basis matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}}$.

(c) If $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find both $[\mathbf{v}]_{\mathcal{C}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

(d) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation so that $T(\mathbf{c}_1) = \mathbf{b}_1 + 2\mathbf{b}_2$ and $T(\mathbf{c}_2) = \mathbf{b}_2$. Find $[T]_{\mathcal{C}}$, the matrix for T relative to $\mathcal{C}$.

5. (a) Are the three polynomials $1 + 2t, t + t^2, 3t^2 + 2t - 4$ a basis for $\mathbb{P}_2$? Explain why or why not.

   (b) Every year 20% of the people in City A move to City B, and 10% of the people in City B move to city A. Suppose that initially, each city has $1,000,000$ people. How many people will live in each city after two years? After a very large number of years?

6. (a) Diagonalize the matrix $A = \begin{bmatrix} 4 & -6 \\ 0 & 1 \end{bmatrix}$.

   (b) Explain how to quickly compute A^{20}. (You don't need to actually do it.)

   (c) Give the solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

7. The matrix

$$A = \begin{bmatrix} 4 & 20 & 8 & 16 & 16 & -3 & 23 \\ 8 & 40 & 5 & 21 & 10 & -2 & 36 \\ 5 & 25 & 2 & 12 & 4 & -1 & 21 \\ 1 & 5 & 2 & 4 & 4 & -1 & 5 \end{bmatrix}$$

can be row reduced to

$$\begin{bmatrix} 1 & 5 & 0 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Give bases for $\operatorname{Row} A$, $\operatorname{Col} A$, and $\operatorname{Nul} A$.

   (b) What is the rank of A, and what are the dimensions of $\operatorname{Row} A^T$, $\operatorname{Col} A^T$, and $\operatorname{Nul} A^T$?

8. (a) Find an LU-decomposition of the matrix $X = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$.

   (b) Use your answer to (a) to find the determinant of X.

9. (a) Find a QR-decomposition of the matrix $Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ onto $\operatorname{Col} Y$?

10. Consider the matrix

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) What are the eigenvalues of H? Find all of the eigenvectors for each eigenvalue.

(b) Is the matrix H diagonalizable? Explain.