Math 310, Lesieutre
Quiz #3
September 16, 2015

Name: ____________________

1. *Determine whether the following vectors are linearly independent:*

$$\begin{bmatrix}1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\end{bmatrix}, \quad \begin{bmatrix}1\\3\end{bmatrix}.$$

Explain your answer.

These vectors are not linearly independent: it is impossible for 3 vectors in $\mathbb{R}^2$ (or 4 in $\mathbb{R}^3$,...) to be linearly independent.

The more popular solution was to actually check it. We want to know if there is a nonzero solution to

$$x_1 \begin{bmatrix}1\\1\end{bmatrix} + x_2 \begin{bmatrix}1\\2\end{bmatrix} + x_3 \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

To figure this out, we need to use row reduction on the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

The variable x_3 is free. This means that there are infinitely many solutions, and in particular that there is some nonzero solution. So the vectors are not independent.

2. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

Find a vector $\mathbf{x}$ whose image under T is $\mathbf{b} = \begin{pmatrix}2\\2\\4\end{pmatrix}$. Is the number of vectors $\mathbf{x}$ with $T(\mathbf{x}) = \mathbf{b}$ finite or infinite?

We are asking if there are any solutions to $A\mathbf{x} = \mathbf{b}$, and if so, whether there are finitely many or infinitely many. This is again a matter for row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 1 & 3 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This is rref. The basic variables are x_1 and x_2, while the variable x_3 is free. If we plug in $x_3 = 0$ (it's a free variable, so we can plug in any value we want and get a solution), we then find $x_1 = 1$ and $x_2 = 1$. So one $\mathbf{x}$ that does the trick is

$$\mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Since there is a free variable, there are infinitely many solutions.

Problem 3 on back!

3. *Let A be the matrix*

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}.$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by $T(\mathbf{x}) = A\mathbf{x}$.

(a) *Plot the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ on the left-hand coordinate axes.*

Done.

(b) *Compute $T(\mathbf{u})$ and $T(\mathbf{v})$, and plot these vectors on the right coordinate axes.*

We get

$$T(\mathbf{u}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix},$$
$$T(\mathbf{v}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(c) *Describe in words how T transforms a vector $\mathbf{x}$.*

It stretches the vector vertically by a factor of 2 while squishing it horizontally by a factor of 1/2.