Math 310, Lesieutre
Quiz #7
October 21, 2015

Name: ____________________

1. *Find a basis for the subspace spanned by the following vectors.*

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

What is the dimension of this subspace?

The span of these vectors is the same as the column space of the matrix we get when we assemble all the vectors into a matrix. To find a basis, we need to figure out which ones are the pivot columns. Row reduction is unavoidable.

$$\begin{bmatrix} -1 & 2 & 0 \\ 3 & -4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

That's echelon form; to find the pivots, we don't need to got all the way to rref. In this case, the pivots are the first and second columns. So our basis is the first and second columns of the original matrix, i.e. the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}.$$

There are two vectors in the basis: that means that the space is two-dimensional.

2. *Find a basis for the nullspace of the matrix*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & 1 \end{bmatrix}$$

What is the dimension of Nul A*?*

We need to find the general solution to $A\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 7 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -9 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

There's a single free variable here, x_3: let's parametrize it using the variable s. We have $x_1 - 9x_3 = 0$, so $x_1 = 9s$. Similarly $x_2 + 4x_4 = 0$, so $x_2 = -4s$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9s \\ -4s \\ s \end{bmatrix} = s \begin{bmatrix} 9 \\ -4 \\ 1 \end{bmatrix}$$

A basis for the nullspace is given by the single vector

$$\begin{bmatrix} 9 \\ -4 \\ 1 \end{bmatrix}.$$

There is a single vector in the basis. That means that $\operatorname{Nul} A$ has dimension 1.

3. *A basis $\mathcal{B}$ for the space $\mathbb{P}_1$ (i.e. polynomials of the form $a + bt$) is given by*

$$\mathbf{b}_1(t) = 2 + t$$
$$\mathbf{b}_2(t) = 1 - t$$

Let $\mathbf{p}(t) = 7 - t$. Compute the coordinate vector $[\mathbf{p}(t)]_{\mathcal{B}}$.

We want to think about this using coordinates for the "normal" basis $\mathcal{E}$ given by 1 and t. We have

$$[\mathbf{b}_1(t)]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ [\mathbf{b}_2(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ [\mathbf{p}(t)]_{\mathcal{E}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix},$$

We want to write $\mathbf{p}(t)$ as a combination of the given polynomials. First we figure out how to do this with the coordinate vectors:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

So we get

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & -3/2 & -9/2 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right].$$

This means $c_1 = 2$ and $c_2 = 3$. The vector is $[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Let's double check that this actually works with the polynomials:

$$2\mathbf{b}_1(t) + 3\mathbf{b}_2(t) = 2(2 + t) + 3(1 - t) = 4 + 2t + 3 - 3t = 7 - t,$$

like we wanted.