Math 310, Lesieutre
Problem set #1
September 2, 2015

Problems for M 8/24:

1.1.2 *Solve the following linear system using elementary row operations on the equations or on the augmented matrix:*

$$2x_1 + 4x_2 = -4$$
$$5x_1 + 7x_2 = 11$$

Let's do this one using row reduction.

$$\left[\begin{array}{cc|c} 2 & 4 & -4 \\ 5 & 7 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 4 & -4 \\ 0 & -3 & 21 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 4 & -4 \\ 0 & 1 & -7 \end{array}\right] \to$$
$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & -7 \end{array}\right]$$

Reading off the linear equations from the matrix, this says $x = 12$ and $y = -7$.

1.1.13 *Same thing, but for the system:*

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2$$

Another row reduction:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

So the solution is $x_1 = 5$, $x_2 = 3$, and $x_3 = -1$.

1.1.17 *Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.*

For there to be a common point of intersection, there must be a point (x_1, x_2) that satisfies all three of these equations. So the question is whether the corresponding system of linear equations is consistent, and that's something we know how to check using row reduction.

$$\left[\begin{array}{rr|r} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -4 & 1 \\ 0 & 7 & -5 \\ -1 & -3 & 4 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{array}\right]$$

This is in echelon form, so we can already tell if the system is consistent or not (we could go all the way to rref if we really wanted to know the specific solution, but there's no need for this problem). There's a row of all 0's, but the system is consistent. This menas that the three lines all do intersect at one point.

Problems for W 8/26:

1.2.2 *Which of the following matrices are in reduced echelon form, which are in echelon form, and which are in neither?*

(a)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This one is in reduced echelon form: the leading entries are 1's, and the pivot entries are the only nonzero entries in their columns.

(b)

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This one is in echelon form, but it isn't in rref. The "1" in the second row and second column is a pivot, but there's another "1" directly above it, which isn't allowed.

(c)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This one isn't even in echelon form: there's a pivot in the top left, but the entry below it isn't a 0.

(d)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is echelon form, but not rref: the leading entry of the second row is a 2 (and there is nonzero stuff above it.)

1.2.3 *Row reduce the following matrix to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The final rref, with pivots marked, is

$$\begin{bmatrix} \textcircled{1} & 0 & -1 & -2 \\ 0 & \textcircled{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are columns 1 and 2.

1.2.4 *Row reduce the following matrix to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.*

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So rref is the following matrix, with pivots circled:

$$\begin{bmatrix} \textcircled{1} & 0 & -1 & 0 \\ 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

The pivot columns and 1, 2, and 4.

Problems for F 8/28:

1.2.7 *Find the general solution of the system whose augmented matrix is*

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$$

First we have to put this thing in rref.

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The pivot variables are x_1 and x_3. The variable x_2 is free. Hence the general solution is

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

1.2.12 *Find the general solution of the system whose augmented matrix is*

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

We have to put this into rref using elementary row operations. Here goes:

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

That wasn't so bad. The pivot variables are x_1 and x_3, and so x_2 and x_4 are free. This means that the general solution is

$$\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$$

(Again, what does this mean? We can plug in anything we want for x_2 and x_4, use this to find x_1 and x_3, and we get a solution. In particular there are infinitely many solutions, because we can pick whatever we want for the free variables.)

1.3.1 *Let*

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

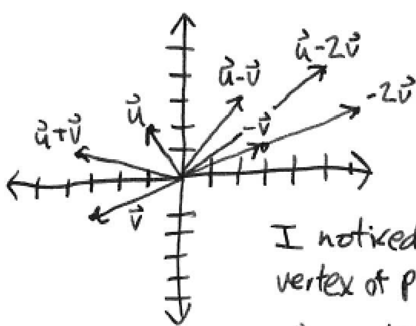
Compute $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{u} - 2\mathbf{v}$.

We have

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}-3\\-1\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$$

$$\mathbf{u}-2\mathbf{v} = \begin{bmatrix}-1\\2\end{bmatrix} - 2\begin{bmatrix}-3\\-1\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}.$$

1.3.3 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be the same vectors as in (1.3.1). Plot the following vectors using arrows on an xy-graph:* $\mathbf{u}$, $\mathbf{v}$, $-\mathbf{v}$, $-2\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, *and* $\mathbf{u}-2\mathbf{v}$. *Notice that* $\mathbf{u}-\mathbf{v}$ *is the vertex of a parallelogram whose other vertices are* $\mathbf{u}$, $\mathbf{0}$, *and* $-\mathbf{v}$.



I noticed that $\vec{u}-\vec{v}$ is vertex of parallelogram whose other vertices are $\vec{u}$, $\vec{O}$, $-\vec{v}$. ✓

1.3.9 *Write a vector equation that is equivalent to the given system of equations.*

$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned}$$

This is the vector equation

$$x_1\begin{bmatrix}0\\4\\-1\end{bmatrix} + x_2\begin{bmatrix}1\\6\\3\end{bmatrix} + x_3\begin{bmatrix}5\\-1\\-8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$